This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

99

1817
ARTES
SCIENTIA
VERITAS
LIBRARY OF THE
UNIVERSITY OF MICHIGAN
TUEBOR
SI QUAERIS PENINSULAM AMOENAM
CIRCUMSPICE

QA
302
R88
1767

AN

INTRODUCTION

TO THE

DOCTRINE

OF

FLUXIONS.

With Thirteen COPPER-PLATES.

By JOHN ROWE.

The THIRD EDITION:
With *Additions* and *Alterations*.

LONDON:
Printed by J. & J. MARCH, on *Tower-Hill*:
And Sold by J. JOHNSON and B. DAVENPORT,
in Pater-noſter-row.
M.DCC.LXVII.

Hist. of sci.
Bowes
8-22-35
30809

Q 9.10-357 [illegible]

TO

WILLIAM DAVY Eſq;

One of His MAJESTY's

SERJEANTS at LAW,

THIS

TRACT

IS INSCRIBED,

By his moſt Obedient

Humble Servant

The AUTHOR.

THE

PREFACE.

OF all the Mathematical Sciences, *The Doctrine of Fluxions* is the moſt extenſive and ſublime. By this, many Difficulties, unſurmountable by any other known Method, are ſolved with uncommon Expedition, Elegance, and Eaſe.

It is a General Way—for determining the *maxima* and *minima* of Quantities; drawing Tangents to Curves, finding their Points of Inflection and Radii of Curvature:—for obtaining the Lengths of curve Lines, the Areas of curvilineal Spaces, the Surfaces and Solidities of concave and convex Bodies: &c. —In a Word, It extends to the inveſtigating the moſt abſtruſe and difficult Problems in the various Branches of mathematical and philoſophical Science.

The *Method of Fluxions* was firſt invented in the Years 1665 and 1666, by that Prince of Mathematicians and Philoſophers the late Sir *Iſaac Newton*, then Mr. *Newton* about 23 Years old *. It was ſoon after communicated to ſome of his Friends; but, he gave no *public* Specimens thereof until the appearance of his immortal *Philoſophiæ Naturalis Principia Mathematica*, printed in the Year 1687: The celebrated *German* therefore, Mr. *Godfrey William Leibnitz* to whom it was hinted in 1676, applied it in the *Acta Eruditorum*, printed at *Leipſic* in 1684, to a few Problems *de Maximis et Minimis* and Tangents to Curves, and claimed the Invention himſelf †. ———Their *Notations* indeed are different; and, Quantities being by both, in Effect, conſidered as produced by continual Increaſe after the ſame manner as Space is deſcribed by a Body in Motion; inſtead of the *Velocity* with which a Quantity varies or flows at any Point or Term of the Time in which it is ſuppoſed to be generated, called by Sir *Iſaac* a *Fluxion*, *Leibnitz* takes the *Increment*, or little Part generated in an indefinitely ſmall Portion of Time, and calls it a *Differential* *.

---

* He was Born *December* 25. 1642, Knighted in 1705, and Died *March* 20. 1726.

† See *Raphſon's Hiſtory* of *Fluxions*, printed in the Year 1715; or, the *Commercium Epiſtolicum*, publiſhed by Order of *the Royal Society* in 1722; wherein, Sir *Iſaac* is fully proved the Original Inventor of this noble and moſt delightful Method.

Several excellent Treatiſes have been publiſhed on the Subject; but, as they appear not calculated to introduce the young and unaſſiſted *Beginner* into this abſtruſe and difficult Science, in order to his underſtanding them a plain and eaſy *Introduction* ſeems to be neceſſary; and for that End the following Sheets are chiefly deſigned.

This Tract is divided into *three* Parts: the *firſt* treats of the *Direct* Method of Fluxions; in which, from the generated Quantity or *Fluent* being given, we find the *Fluxion*: and the *ſecond* of the *Inverſe* Method; wherein, from the *Fluxion* being known, we find the *Fluent* *; the *third* contains *miſcellaneous Queſtions* with their *incremental* and *fluxional Solutions*; which could not, with propriety, be inſerted in the former Parts; and to ſome of which there was occaſion to infer.

---

* *Leibnitz* denotes the *Differential* of any variable quantity x by dx: and Sir *Iſaac*, generally, for it's *Fluxion* writes $\dot{x}$; but in his *Principia* flowing quantities are expreſſed by the capital letters, A, B, &c. and their *Fluxions* or *Increments* by the correſponding ſmall letters *a*, *b*, &c.

In this *Third Edition* † are many *Additions* and *Alterations*. The *Conſtructions* are, in general, *New* ‡. In a Word, It contains, perhaps, a variety of Things not to be found in any other Tract on the Subject.

In order to a thorough underſtanding of this *Introduction*, it is requiſite that the Learner be well acquainted with Arithmetic, Algebra, Geometry, Plane-Trigonometry, Conic-Sections, and the Nature of Logarithms. But, as geometrical and algebraical Treatiſes, in general, give not the Deſcriptions, and from thence the deduction of the Properties, of ſome Curves to be found in the following Sheets; nor the Methods of reducing Quantities into Infinite Series, and of Noting their Powers and Roots, neceſſary to be uſed in *fluxional* Tracts; therefore, theſe Deficiencies are herein ſupplied, though they do not immediately relate to the Buſineſs in Hand.

---

* The *Direct* Method of Fluxions, as delivered by *Leibnitz*, is, by *Foreigners*, called *Calculus Differentialis*; and the *Inverſe* Method, *Calculus Integralis*.

† The *Firſt Edition* was printed in the Year 1751; and the *Second*, with *Alterations* and *Additions*, in 1757.—Cuts.

‡ Thoſe in *art.* 35, 39, 78, and 86, may be ſeen in other Books.

JOHN ROWE.

January 8. 1767.

THE CONTENTS.

PART I.

CHAP. I. *Of the Principles of* Fluxions, *and of the New Notation in* Algebra Page 1
CHAP. II. *Of finding the Fluxion of a given Fluent* 13
CHAP. III. *Of drawing Tangents to Curves* 28
CHAP. IV. *Of finding the* Maxima *and* Minima, *or* Greateſt *and* Leaſt *of variable Quantities* 56
CHAP. V. *Of finding the Points of Inflection in Curves* 87
CHAP. VI. *Of finding the Radius of Curvature* 101
CHAP. VII. *Of finding the Nature of the Evolute of a given Involute Curve* 124

PART II.

CHAP. I. *Of Infinite Series* Page 133
CHAP. II. *Of finding the Fluent of a given Fluxion* 148
CHAP. III. *Of finding the Length of a Curve Line* 158
CHAP. IV. *Of finding the Areas of Curvilineal Spaces* 168
CHAP. V. *Of finding the Convex Superficies of Solids* 175
CHAP. VI. *Of finding the Contents of Solids* 180

PART III.

MISCELLANEOUS QUESTIONS, *with their Incremental and Fluxional* SOLUTIONS 189

AN

AN INTRODUCTION TO THE DOCTRINE OF FLUXIONS.

PART I.

CHAP. I.

Of the Principles of Fluxions, *and of the New Notation in* Algebra.

1. IN this Doctrine, Quantities are ſuppoſed to be generated by continual increaſe, after the manner of a Space which a body in motion deſcribes.

Thus, a Line is ſuppoſed to be generated by a Point in motion, a Superficies by a Line, and a Solid by a Superficies.

2. THE Velocity with which a Quantity flows, or is generated, at any particular point or term of it, is called the *Fluxion* of that Quantity at that point or term.

Fig. 1. 2. Thus, if we ſuppoſe the indefinite right line A Z to move with a parallel motion along the axis A X, or, ſo as always to be parallel to it's firſt ſituation; and, at the ſame time, a point to move from A along the ſaid line A Z ſo as to generate or always to be in the curve A Y: Then, the Velocity with which the end or point A of the line A Z arrives at any point C, or, which is the ſame, the Velocity with which the axis flows or is generated at any particular point C, is called the Fluxion of the axis at that point; and the Velocity with which the point moves along the line A Z at any point B, that is, the Velocity with which the ordinate flows or increaſes at any point B, is called the Fluxion of the ordinate at that point; alſo, the Velocity with which the point generates or moves along the curve at any point B, is called the Fluxion of the curve at that point; likewiſe, the Velocity, or degree of quickneſs, with which the curvilineal ſpace A C B flows or is generated by the line A Z at any term

C B, is called the Fluxion of the ſaid curvilineal ſpace at that term.

3. Now, if the Velocity with which any quantity flows, or is generated, be at every point or term the ſame; that is, if it be neither accelerated nor retarded; the Fluxion of it will likewiſe be at every point or term the ſame. But if this Velocity be continually increaſed or diminiſhed, then there will be a certain degree of Velocity, or Fluxion, peculiar to every point or term of the thing deſcribed: And the Velocity wherewith the ſaid velocity, at any point or term, is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *Second Fluxion.* And, again, if this acceleration or retardation be not uniform, but is continually varying; or the velocity with which the quantity flows does not uniformly increaſe or decreaſe; then, the Velocity, or degree of ſwiftneſs, with which this acceleration or retardation either increaſes or decreaſes, is called the *Third Fluxion*. And ſo on.

4. THE indefinitely ſmall Increaſe of a Quantity, generated in an indefinitely ſmall particle of time, is called the *Increment* of that Quantity.

Fig. Thus, if we ſuppoſe *bc* indefinitely
3. near and parallel to the ordinate BC; and
4. B*n* parallel to the abſciſs AC; then, C*c*, or
it's equal B*n*, is called the Increment of the abſciſs AC; *nb* the Increment of the ordinate CB; B*b* the Increment of the curve AB; and CB*bc* the Increment of the curvilineal ſpace ACB.

5. Now, if *ed* be ſuppoſed indefinitely near and parallel to *bc*, and *br* equal and parallel to B*n* or *cd*; then, the Difference between *nb* and *re* is called the *Increment of the Increment*, or the *Second Increment*; that is, the *Increment* of *nb*, or *Second Increment* of CB. And, again, if *sf* be ſuppoſed indefinitely near and parallel to *ed*, and *et* equal and parallel to *br* or *df*; then, the Difference between the Second Increment and that of *re* and *ts*, is called the *Third Increment* of CB. And ſo on.

6. *Note.* When a Quantity, inſtead of increaſing, is continually diminiſhed; then, the indefinitely ſmall Particles by which it is leſſened, are not, properly, called *Increments*, but *Decrements*. And both *Increments* and *Decrements* are ſometimes called *Moments*.

7. Now, if we ſuppoſe the abſciſs, A C, to flow on with an uniform motion, or equal parts of it to be generated or deſcribed in equal times; it's Increment will accurately expreſs or be exactly as it's Fluxion; ſince velocity is always expreſſed by, or is as the ſpace uniformly deſcribed in a given time: And therefore, if the Curve B *b* did exactly coincide with the Tangent or right line T B G, it is evident, that then, the Increments B *b* and *b n* would likewiſe be deſcribed with uniform motions, and the ſame degrees of velocity with which the curve and ordinate reſpectively flow at the point B; that is, the Increments B *b* and *b n* would then be accurately as the Fluxions of the curve and ordinate at the point B. But, ſince not two points of the Curve are coincident with the Tangent, and conſequently the velocities with which the Increments are generated are continually varying in every point; therefore, the Increments and Fluxions are not in an exact proportion to each other; or, the Increments do not accurately meaſure the Velocities or Fluxions with which they begin to be generated. However, as the point *b* is continually nearer to a coincidence with the Tangent G B the nearer it

approaches the point of contact B; ſo therefore, if we conceive the ordinate *cb* to move back until it coincides with C B; then, the very firſt moment before it's coincidence, the curve B *b* and right line B G will be infinitely or rather indefinitely near to a coincidence with each other; and conſequently, in that caſe, the Increments B *b* and *bn* will come indefinitely near to meaſure the Fluxions of the curve and ordinate, or the Velocities with which they flow at the point B: Or, becauſe the particles of time in which any Increments are generated are ſuppoſed to be indefinitely ſmall, and conſequently, the acceleration or retardation of the Velocities with which they are generated muſt be ſo too; therefore, they are indefinitely near in proportion to the Fluxions of the quantities of which they are Increments: But, when *Ratios*, from that of Equality, are but indefinitely little, or leſs than can be aſſigned, they may be conſidered as Equal *. Hence therefore, the Increments may be taken as proportional to, or for the Fluxions in all operations; and, on the contrary, the Fluxions for the Increments.

---

* This was allowed by the ancient Geometricians, *Euclid*, *Archimedes*, &c.

8. THOSE quantities which are ſuppoſed to flow, or to be generated by continual increaſe, as the abſciſs and ordinate of a curve, are called *Fluents*, and *variable* or *flowing* quantities: And thoſe which neither increaſe nor decreaſe, or admit of no variation, as the parameter of a conic-ſection and the diameter of a circle, are called *fixed*, *given*, and *invariable* quantities.

9. THE Beginning of the Alphabet, viz. a, b, c, *&c.* is uſed to expreſs Invariable quantities; and the End of it, viz. z, y, x, *&c.* Variable or Flowing quantities.

10. THE Fluxion of any variable quantity x, is denoted by $\dot{x}$; it's Second Fluxion, or the Fluxion of $\dot{x}$, by $\ddot{x}$; it's Third Fluxion, or the Fluxion of $\ddot{x}$, by $\dddot{x}$; and ſo on. Alſo, the Moment Increment or Decrement of x, is denoted by x'; it's Second Moment Increment or Decrement, or the Moment Increment or Decrement of x', by x''; and ſo on.

11. THE Fluxions and Moments of Invariable quantities, viz. of a, b, c, *&c.* are evidently $= 0$.

12. THOSE Fluents which are generated in the ſame time, or in equal times, or which begin together and end together, are called

contemporary Fluents; and the Fluxions of theſe contemporary Fluents are called *contemporary* Fluxions: Now, it is evident, if two or more of theſe contemporary Fluents are always equal, or in any invariable ratio to each other, that their contemporary Fluxions will likewiſe be equal, or in the ſame proportion: And that, on the contrary, if two or more of theſe contemporary Fluxions are always equal, or in any invariable ratio to each other, their contemporary Fluents will likewiſe be equal, or in the ſame proportion.

WE come now to find the *Fluxions* of *Fluents*, or the Velocities with which Flowing quantities increaſe or decreaſe at any points or terms aſſigned: the Buſineſs of which is called the *Direct* Method of Fluxions. But, before we proceed, it may, perhaps, be Neceſſary to treat of what is called

THE NEW METHOD OF NOTATION IN ALGEBRA*.

13. IN Surds, the Index ſhewing the Height of the Power to which any given

---

* This was invented by the late celebrated Dr. *Wallis*, and firſt publiſhed in his *Arithmetica Infinitorum* in the Year 1656.

quantity is to be raiſed, is here placed as the Numerator of a Fraction, whoſe Denominator is the radical ſign or Index ſhewing the Root to be extracted *.

Thus, $\sqrt[3]{x^2}$ is expreſſed by $x^{\frac{2}{3}}$, and $\sqrt{\overline{a+x}}$ by $\overline{a+x}|^{\frac{1}{2}}$.

Alſo, $\frac{1}{\sqrt[3]{x^2}}$ is expreſſed by $\frac{1}{x^{\frac{2}{3}}}$ or $x^{-\frac{2}{3}}$, and $\frac{1}{\sqrt[3]{a+x}}$ by $\frac{1}{\overline{a+x}|^{\frac{1}{3}}}$ or $\overline{a+x}|^{-\frac{1}{3}}$. §

Now, in any geometrical progreſſion, whoſe firſt term is *unity*, or 1; if we take an arithmetical progreſſion, whoſe firſt term is 0, and whoſe ſecond term, or common difference, is the Index of the quantity in the ſecond term of the geometrical progreſſion; then, the number in any term of the arithmetical progreſſion will be the Index of the quantity in the correſponding term of the

---

[*] This Fraction is called the *Index* of the Power, and the given quantity itſelf the *Root* of the Power.

[§] The propriety of uſing *negative* Indices is evident; for, to divide any Power of x by x, is only to leſſen the Index of the Power by 1; or, to ſubtract the Index of the Denominator from that of the Numerator. Thus, $\frac{x^3}{x^1}$ is $= x^2$; $\frac{x^2}{x^1}$ is $= x^1$ or x; $\frac{x^1}{x^1}$ is $= x^0 = 1$; $\frac{x^0}{x^1}$ is $= x^{0-1}$, that is, $\frac{1}{x}$ is $= x^{-1}$; &c.

the geometrical progreſſion: And therefore, the arithmetical mean between the numbers in any two terms of the arithmetical progreſſion will be the Index of the geometrical mean between the quantities in the two correſponding terms of the geometrical progreſſion.

Thus, in the geometrical progreſſion $1. x^{\frac{1}{2}}. x^{1}. x^{\frac{3}{2}}. x^{2}. x^{\frac{5}{2}}. x^{3}.$ *&c.* where the common multiplier is $x^{\frac{1}{2}}$; if we take the arithmetical progreſſion $0. \frac{1}{2}. 1. \frac{3}{2}. 2. \frac{5}{2}. 3.$ *&c.* wherein the common difference, or quantity added, is $\frac{1}{2}$; the number in either term of the arithmetical progreſſion is the Index of the quantity in the correſponding term of the geometrical progreſſion: And, the arithmetical mean between the numbers in any two terms of the arithmetical progreſſion is the Index of the geometrical mean between the quantities in the two correſponding terms of the geometrical progreſſion. For inſtance, $\frac{3}{2}$ the 4th term of the arithmetical progreſſion, is the Index of $x^{\frac{3}{2}}$, the 4th term of the geometrical progreſſion: And, the arithmetical mean between $\frac{1}{2}$ and $\frac{5}{2}$, the 2d and 6th terms of the arithmetical progreſſion, which is $\frac{3}{2}$, is the Index of the geometrical mean between $x^{\frac{1}{2}}$ and $x^{\frac{5}{2}}$, the ſame two terms of the geometrical progreſſion, which is $x^{\frac{3}{2}}$.

Or, in the deſcending geometrical progreſſion, or ſeries, $1.\frac{1}{x^{\frac{1}{2}}}.\frac{1}{x}.\frac{1}{x^{\frac{3}{2}}}.\frac{1}{x^{2}}.\frac{1}{x^{\frac{5}{2}}}.\frac{1}{x^{3}}.$ &c. that is, $1.x^{-\frac{1}{2}}.x^{-1}.x^{-\frac{3}{2}}.x^{-2}.x^{-\frac{5}{2}}.x^{-3}.$ &c. where the common diviſor is $x^{\frac{1}{2}}$; if we take the arithmetical progreſſion, or ſeries, $0.-\frac{1}{2}.-1.-\frac{3}{2}.-2.-\frac{5}{2}.-3.$&c. wherein the common difference, or quantity ſubtracted, is $\frac{1}{2}$; the number in any term of the arithmetical ſeries, is the Index of the quantity in the correſponding term of the geometrical ſeries: thus, $-\frac{3}{2}$, the 4th term of the arithmetical ſeries, is the Index of $x^{-\frac{3}{2}}$, the ſame term of the geometrical ſeries. And, the arithmetical mean between any two terms of the arithmetical ſeries, is the Index of the geometrical mean between the ſame two terms of the geometrical ſeries: as, for inſtance, the arithmetical mean between $-\frac{1}{2}$ and $-\frac{5}{2}$, the 2d and 6th terms of the arithmetical ſeries, which is $-\frac{3}{2}$, is the Index

of

of the geometrical mean between $x^{\overline{-\frac{1}{2}}}$ and $x^{\overline{-\frac{5}{2}}}$, the 2d and 6th terms of the geometrical ſeries, which is $x^{\overline{-\frac{3}{2}}}$.

HENCE we may obſerve, that, in the Indices of Powers, Addition has the effect of Multiplication on the reſpective Roots, and Multiplication of Involution; and, *e contra*, Subtraction of Diviſion, and Diviſion of Evolution: or that, in a word, Indices of Powers are entirely Logarithmical with regard to their Roots.

So that, $x^{2}\times x^{\frac{3}{2}}$ is $=x^{2+\frac{3}{2}}=x^{\frac{7}{2}}$, $\left.\overline{x^{3}}\right|^{2}$ is $=x^{3\times 2}=x^{6}$, $\frac{x^{5}}{x^{3}}$ is $=x^{5-3}=x^{2}$, $\left.\overline{x^{4}}\right|^{\frac{1}{2}}$ is $=x^{\frac{4}{2}}=x^{2}$; and $x^{2}\times x^{\overline{-\frac{3}{2}}}$ is $=x^{2\overline{-\frac{3}{2}}}=x^{\frac{1}{2}}$, $\left.\overline{x^{2}}\right|^{-3}$ is $=x^{2\times -3}=x^{-6}$, $\left.\overline{x^{4}}\right|^{-\frac{1}{2}}$ is $=x^{4\times -\frac{1}{2}}=x^{-2}$: or, Univerſally, $x^{m}\times x^{n}$ is $=x^{m+n}$, $\left.\overline{x^{m}}\right|^{n}$ is $=x^{mn}$, $\frac{x^{m}}{x^{n}}$ is $=x^{m-n}$, and $\left.\overline{x^{m}}\right|^{\frac{1}{n}}$ is $=x^{\frac{m}{n}}$; where note, *m* and *n* repreſent any affirmative or negative whole-numbers or fractions whatever [*].

---

[*] Theſe Indefinite Indices were introduced by the great Inventor of Fluxions.

IF what has been ſaid be duly conſidered, no difficulty in this *New Notation* will occur; the knowledge of which is abſolutely neceſſary, in order to the well underſtanding the following pages.

CHAP. II.

Of finding the Fluxion of a given Fluent.

RULE I.

14. To find the Fluxion of a Simple Fluent, or, of that wherein there is but One variable letter or flowing quantity.

MARK the variable letter or flowing quantity with a dot over it: and you will have the Fluxion required:

Thus, the Fluxion of ax is $= a\dot{x}$.

For, if we ſuppoſe the variable rectangle A B to be generated by the given line C B ſetting out from the ſituation A D and moving along with a parallel motion between the parallel and indefinite ſides A C and D B; it is evident, the Velocity with which the rectangle flows is equal to the generating line C B drawn into the Velocity with which the point C generates or moves along the line A C: *Fig.* 5.

AC: that is, the Fluxion of the rectangle AB is equal to the invariable line CB drawn into the Fluxion of the flowing or variable line AC. Therefore, if AD or CB $= a$, and AC or DB $= x$; then, the Fluxion of the rectangle ax will be $= a \times \dot{x} = a\dot{x}$.

RULE II.

15. To find the Fluxion of the Product of two or more flowing quantities drawn into each other.

MULTIPLY the Fluxion of each quantity ſeparately, by the other or the product of the reſt of the quantities: and the Sum of theſe products will be the Fluxion required *.

Thus, the Fluxion of xy is $= \dot{x}y + x\dot{y}$; the Fluxion of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$; and the Fluxion of $vxyz$ is $= \dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$.

Fig. 6. 1°. That the Fluxion of xy is $= \dot{x}y + x\dot{y}$, may thus be proved. Suppoſe a line, coincident with the indefinite right line A F, to move with a parallel motion along the indefinite right line A E perpendicular to A F; and at the ſame time, another, coincident with the line A E, to move with a parallel motion along the line A F; and that the ſaid two lines move with ſuch different degrees of velocity as that their points of interſection be always in the curve A I. Through any point B in the curve, draw C H parallel to A F, and D G parallel to A E. Then, by the line moving along the line A E will the curvilineal ſpace A E I and rectangle D E be generated; and by the line moving along the line A F the curvilineal ſpace A F I and rectangle C F will be generated. Now, before the line moving along the line A E arrives at the ſituation C H, it is evident, that, the curvilineal ſpace A E I will increaſe ſlower, or, flow with a leſs degree of velocity, than the rectangle D E, and afterwards faſter, or with a greater degree of velocity; therefore, at the term C B, they will flow or increaſe with one and the ſame degree of velocity: So likewiſe, the curvilineal ſpace A F I will flow or increaſe ſlower, or with a leſs degree of velocity, than the rectangle C F, before the generating line, moving along the line A F, comes to the ſituation D G; and, afterwards, faſter, or with a greater degree of velocity; and therefore, at the term DB, they will increaſe or flow with an equal degree of velocity. That is, the Fluxion of the curvilineal ſpace A E I is, at the term CB, equal to the Fluxion of the rectangle D E at the ſaid term C B; and, the Fluxion of the curvilineal ſpace A F I is, at the term D B, equal to the Fluxion of the rectangle C F at the ſame term D B. But, (*art.* 14.) the Fluxion of the rectangle D E, at the term C B, is equal to C B drawn into the Fluxion of AC; and the Fluxion of the rectangle C F, at the term DB, is equal to D B drawn into the Fluxion of AD. Hence therefore, the Fluxion of the flowing rectangle A C B D (or of the ſum of the two curvilineal ſpaces ACB and ADB) is equal to CB drawn into the Fluxion of A C, added to D B drawn into the Fluxion of A D: that is, if we put AC or DB $= x$, and AD or CB $= y$, the Fluxion of the rectangle xy will be $= \dot{x}y + x\dot{y}$.

---

* The Common Methods of proving the truth of this Rule, which are by the aid of Increments, were ſmartly attacked by the late acute Dr. *Berkeley*, Biſhop of *Cloyne*, in a Pamphlet called *The Analyſt*, printed in the Year 1734: But, his Objections, it is preſumed, are by no means applicable to the demonſtrations here given: On the contrary, it is not doubted, but the reaſoning here advanced will be allowed to be ſcientific, fair and concluſive.

2°. And, that the Fluxion of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$, may thus be proved. Let the length, breadth, and depth, of a parallelopipedon, be repreſented by x, y, and z, reſpectively. Now, this parallelopipedon will be equal to three pyramids, whoſe baſes are xy, xz, and yz, and altitudes z, y, and x, reſpectively*: and therefore, (*art.* 12.) the Sum of the Fluxions of theſe pyramids will be equal to the Fluxion of the parallelopipedon xyz. Let either of the pyramids be repreſented by AEI or ACB, which ſuppoſe to be generated by the variable plane AF moving with a parallel motion along the indefinite right line AE; and, at the ſame time, let the parallelopipedon ADGE (whoſe face AD or EG is equal and ſimilar to the baſe of the pyramid at the term CB) be generated by the plane AD always coincident with the plane AF. Now, it is plain, that, the pyramid will increaſe ſlower, or flow with a leſs degree of velocity, than the parallelopipedon, before the generating planes arrive at the term CB, and afterwards, faſter, or with a greater degree of velocity; therefore, at the ſaid term CB, they will flow, or be generated, with the ſame or an equal degree of velocity: but, it is likewiſe plain, that, the velocity with which the parallelopipedon is generated, is equal to it's face AD or CB drawn into the *Fig.* 7.

---

* 7 E. 12. Corol. 1.

the velocity of it's motion along the line or ſide AE; therefore, the velocity with which the pyramid is generated, is, at the term C B, equal to it's baſe C B drawn into the velocity of it's motion, at the point C, along the ſide AE: that is, the Fluxion of the pyramid A C B is equal to it's baſe C B drawn into the Fluxion of it's altitude A C. Hence therefore, when the baſe is xy and altitude z, the Fluxion of the pyramid is $= xy \times \dot{z} = xy\dot{z}$; and when the baſe is xz and altitude y, the Fluxion of it is $= xz \times \dot{y} = x\dot{y}z$; and, laſtly, when the baſe is yz and altitude x, it's Fluxion is $= yz \times \dot{x} = \dot{x}yz$. Conſequently, $\dot{x}yz + x\dot{y}z + xy\dot{z}$ (which is the Sum of the Fluxions of the three pyramids,) is = the Fluxion of the parallelopipedon xyz.

3°. And, hence, that the Fluxion of $vxyz$ is $= \dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$, may be thus proved. Put $xyz = u$; then $vxyz = vu$: therefore, (*art.* 12.) the Fluxion of u is = the Fluxion of xyz, and the Fluxion of vu is = the Fluxion of $vxyz$; that is, $\dot{u} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, and $\dot{v}u + v\dot{u}$ = the Fluxion of $vxyz$. Wherefore, by reſtitution, or writing xyz for u, and $\dot{x}yz + x\dot{y}z + xy\dot{z}$ for $\dot{u}$, we have $\dot{v}u+v\dot{u}=\dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}=$ the Fluxion of $vxyz$.

HENCE,

16. The Fluxion of xx is $=\dot{x}x+x\dot{x}$; the Fluxion of xxx is $=\dot{x}xx+x\dot{x}x+xx\dot{x}$; the Fluxion of $xxxx$ is $=\dot{x}xxx+x\dot{x}xx+xx\dot{x}x+xxx\dot{x}$: that is, the Fluxion of x^2 is $=2x\dot{x}$; the Fluxion of x^3 is $=3x^2\dot{x}$; the Fluxion of x^4 is $=4x^3\dot{x}$. And, if m repreſents any affirmative or poſitive whole-number, the Fluxionof x^m will be $=mx^{m-1}\dot{x}$.

RULE III.

17. To find the Fluxion of a Fraction.

MULTIPLY the Denominator into the Fluxion of theNumerator; from the product of which, ſubtract the Numerator drawn into the Fluxion of the Denominator; then, divide the Remainder by the Square of the Denominator: and you will have the Fluxion of the Fraction required.

Thus, the Fluxion of $\frac{x}{y}$ is $=\frac{\dot{x}y-x\dot{y}}{y^2}$.

For, put $z=\frac{x}{y}$; then $yz=x$; and the Fluxion of this equation (*art.* 12 and 15.) is $\dot{y}z+y\dot{z}=\dot{x}$: therefore, by tranſpoſition,

$y\dot{z}$

$y\dot{z} = \dot{x} - \dot{y}z$, and, by diviſion, $\dot{z} = \frac{\dot{x} - \dot{y}z}{y}$; that is, by reſtitution, or writing $\frac{x}{y}$ for z it's equal, $\dot{z} = \frac{\dot{x} - \dot{y} \times \frac{x}{y}}{y} = \frac{\dot{x}y - x\dot{y}}{y^2} =$ the Fluxion of $\frac{x}{y}$.

Alſo, the Fluxion of $\frac{1}{x}$ is $= \frac{-\dot{x}}{x^2}$; the Fluxion of $\frac{1}{x^2}$ is $= \frac{-2x\dot{x}}{x^4}$; the Fluxion of $\frac{1}{x^3}$ is $= \frac{-3x^2\dot{x}}{x^6}$; and the Fluxion of $\frac{1}{x^4}$ is $= \frac{-4x^3\dot{x}}{x^8}$. For,

1°. Put $z = \frac{1}{x}$; then $xz = 1$, and the Fluxion of xz will be $=$ the Fluxion of 1, which (*art.* 11,) is $= 0$; that is, (*art.* 15.) $\dot{x}z + x\dot{z} = 0$: therefore, $x\dot{z} = -\dot{x}z$, and $\dot{z} = \frac{-\dot{x}z}{x}$; or, by writing for z it's equal $\frac{1}{x}$, $\dot{z} = \frac{-\dot{x}}{x^2} =$ the Fluxion of $\frac{1}{x}$.

2°. Put $z = \frac{1}{x^2}$; then, $zx^2 = 1$; and the Fluxion of this equation (*art.* 11. 15. and 16.)

16.) is $\dot{z}x^2+2zx\dot{x}=0$: therefore, $\dot{z}x^2=-2zx\dot{x}$, and $\dot{z}=\frac{-2zx\dot{x}}{x^2}$; or, by writing $\frac{1}{x^2}$ for it's Value z, $\dot{z}=\frac{-2x\dot{x}}{x^4}=\frac{-2\dot{x}}{x^3}=$ the Fluxion of $\frac{1}{x^2}$.

3°. Put $z=\frac{1}{x^3}$; then, $zx^3=1$; and the Fluxion of this Equation (*art.* 11. 15. and 16.) is $\dot{z}x^3+3zx^2\dot{x}=0$: therefore, $\dot{z}x^3=-3zx^2\dot{x}$, and $\dot{z}=\frac{-3zx^2\dot{x}}{x^3}$; or, by writing $\frac{1}{x^3}$ for z it's value, $\dot{z}=\frac{-3x^2\dot{x}}{x^6}=\frac{-3\dot{x}}{x^4}=$ the Fluxion of $\frac{1}{x^3}$.

4°. Put $z=\frac{1}{x^4}$; then, $zx^4=1$; the Fluxion of which Equation (*art.* 11. 15. and 16.) is $\dot{z}x^4+4zx^3\dot{x}=0$: therefore $\dot{z}x^4=-4zx^3\dot{x}$, and $\dot{z}=\frac{-4zx^3\dot{x}}{x^4}$; that is, by reſtitution, or writing $\frac{1}{x^4}$ for z, $\dot{z}=\frac{-4x^3\dot{x}}{x^8}=\frac{-4\dot{x}}{x^5}=$ the Fluxion of $\frac{1}{x^4}$.

BUT, by the *new* Method of Notation in Algebra, (*art.* 13.) $\frac{1}{x}$ is $=x^{-1}$, $\frac{1}{x^2}$ is $=x^{-2}$, $\frac{1}{x^3}$ is $=x^{-3}$, $\frac{1}{x^4}$ is $=x^{-4}$: alſo, $\frac{-\dot{x}}{x^2}$ is $= -x^{-2}\dot{x}$, $\frac{-2\dot{x}}{x^3}$ is $= -2x^{-3}\dot{x}$, $\frac{-3\dot{x}}{x^4}$ is $= -3x^{-4}\dot{x}$, $\frac{-4\dot{x}}{x^5}$ is $= -4x^{-5}\dot{x}$.

HENCE,

18. The Fluxion of x^{-1} is $= -x^{-2}\dot{x}$, the Fluxion of x^{-2} is $= -2x^{-3}\dot{x}$, the Fluxion of x^{-3} is $= -3x^{-4}\dot{x}$, the Fluxion of x^{-4} is $= -4x^{-5}\dot{x}$. And, if *m* repreſents any negative whole-number, the Fluxion of x^m will be $= mx^{m-1}\dot{x}$.

RULE IV.

19. To find the Fluxion of an expreſſion compounded of different terms or quantities connected together by the Signs + and —.

FIND the Fluxion of each term by the preceding Rules; which connect together by the Signs of the reſpective terms: and you will have the Fluxion required.

Thus, the Fluxion of $ax+xy-\frac{x}{y}$ is $=$ $a\dot{x}+\dot{x}y+x\dot{y}-\frac{\dot{x}y-x\dot{y}}{y^2}$.

For, put $ax+xy-\frac{x}{y}=v$; then, $ax+xy=v+\frac{x}{y}$. Now, (*art.* 12.) it is evident, that, the Sum of the Fluxions of ax and xy muſt always be equal to the Sum of the Fluxions of v and $\frac{x}{y}$; that is, $a\dot{x}+\dot{x}y+x\dot{y}=\dot{v}+\frac{\dot{x}y-x\dot{y}}{y^2}$: therefore, by tranſpoſition, $a\dot{x}+\dot{x}y+x\dot{y}-\frac{\dot{x}y-x\dot{y}}{y^2}=\dot{v}$ $=$ the Fluxion of $ax+xy-\frac{x}{y}$.

RULE V.

20. To find the Fluxion of any Power of a given Fluent; whether the Index be integral or fractional, affirmative or negative.

MULTIPLY the expreſſion by the Index of the Power; then, ſubtract 1 from the ſaid Index, and multiply the reſulting expreſſion by the Fluxion of the given Fluent, or of the Root of the Power: and you will have the Fluxion required.

Thus, the Fluxion of $\overline{x+x^2}\big|^3$ is $=3\times\overline{x+x^2}\big|^{3-1}\times\overline{\dot{x}+2x\dot{x}}=\overline{x+x^2}\big|^2\times\overline{3\dot{x}+6x\dot{x}}$;

the Fluxion of $\overline{2ax - x^2}|^{\frac{1}{2}}$ is $= \frac{1}{2} \times \overline{2ax - x^2}|^{\frac{1}{2} - 1} \times \overline{2a\dot{x} - 2x\dot{x}} = \overline{2ax - x^2}|^{-\frac{1}{2}} \times \overline{a\dot{x} - x\dot{x}} =$ (becauſe, *art.* 13. $\overline{2ax - x^2}|^{-\frac{1}{2}}$ is $= \frac{1}{\overline{2ax - x^2}|^{\frac{1}{2}}}$,) $\frac{a\dot{x} - x\dot{x}}{\overline{2ax - x^2}|^{\frac{1}{2}}}$; the Fluxion of $\overline{a + x}|^{-2}$ is $= -2 \times \overline{a + x}|^{-2-1} \times \dot{x} = \overline{a + x}|^{-3} \times -2\dot{x} = \frac{-2\dot{x}}{\overline{a + x}|^{3}}$; the Fluxion of $\overline{x - a}|^{-\frac{2}{3}}$ is $= -\frac{2}{3} \times \overline{x - a}|^{-\frac{2}{3} - 1} \times \dot{x} = -\frac{2}{3} \times \overline{x - a}|^{-\frac{5}{3}} \times \dot{x} = \frac{1}{\overline{x - a}|^{\frac{5}{3}}} \times -\frac{2}{3}\dot{x} = \frac{-2\dot{x}}{3 \times \overline{x - a}|^{\frac{5}{3}}}$. And, Univerſally, the Fluxion of $x^{\frac{m}{n}}$ is $= \frac{m}{n} \times x^{\frac{m}{n} - 1} \times \dot{x} = \frac{m}{n} x^{\frac{m}{n} - 1} \dot{x} = \frac{m}{n} x^{\frac{m-n}{n}} \dot{x}$; where, *note*, either *m* or *n*, or both *m* and *n*, may be either integral or fractional, affirmative or negative; and conſequently, the Index $\frac{m}{n}$, expreſſes, or repreſents, any affirmative or negative whole-number or fraction whatſoever. For, by *art.* 16 and 18, if *m* repreſents any affirmative or negative whole-number, the Fluxion of x^m will be $= m x^{m-1} \dot{x}$; and therefore, if we ſuppoſe $y = x^{\frac{m}{n}}$, or, which is the ſame, $y^n = x^m$, and n to be any affirmative or negative whole-number likewiſe, by *art.* 12. the Fluxion of y^n will be $=$ the Fluxion of x^m; that is, $n y^{n-1} \dot{y} = m x^{m-1} \dot{x}$; which equation divided by $n y^{n-1}$ makes $\dot{y} = \frac{m x^{m-1} \dot{x}}{n y^{n-1}}$, that is, by writing $x^{\frac{m}{n}}$ for y it's value, $\dot{y} = \frac{m x^{m-1} \dot{x}}{n \times \overline{x^{\frac{m}{n}}}^{\,n-1}} = \frac{m}{n} \times \frac{x^{m-1} \dot{x}}{x^{\frac{mn-m}{n}}} = \frac{m}{n} \times x^{m-1-\frac{mn-m}{n}} \dot{x}$ $= \frac{m}{n} x^{\frac{m-n}{n}} \dot{x} =$ the Fluxion of $x^{\frac{m}{n}}$: and that either m or n, or both m and n, may repreſent any fractions as well as whole-numbers, is plain, ſince $\frac{m}{n}$ repreſents the quotient of any whole-number divided by another, and may be taken for a New m or n; and ſo on *ad infinitum*: therefore, Univerſally, &c.

RULE VI.

21. To find the Fluxion of a Logarithm.

THE Fluxion of the Hyperbolic Logarithm of any quantity, is equal to the Fluxion of that quantity, divided by the quantity itſelf.

Thus, the Fluxion of the Hyp. Log. of x is $=\frac{\dot{x}}{x}$. (See *Part* 3. *Queſt.* 8.)

Now, as the Hyp. Log. of 10 (*viz.* 2.30258 *&c.* See *Part* 3. *Queſt.* 9.) is to the Common Log. of 10 (*viz.* 1.) ſo is the Hyp. Log. of any number, x, to the Common Log. of the ſame number, x; that is, if we put $L = 2.30258$ *&c.* as $L : 1 :: $ Hyp. Log. of x : Common Log. of x. Therefore, (*art.* 12.) $L : 1 ::$ Fluxion of the Hyp. Log. of x : Fluxion of the Common Log. of x.

HENCE, $L : 1 :: \frac{\dot{x}}{x} : \frac{\dot{x}}{Lx} =$ the Fluxion of the Common Log. of x; or, becauſe $\frac{1}{L} =$ 0.43429 *&c.* if we put $\frac{1}{L} = M$, then the Fluxion of the Common Log. of x (*viz.* $\frac{\dot{x}}{Lx}$) will be $= \frac{\dot{x}}{x} \times M$; that is, the Fluxion of the Hyp. Log. of any number multiplied by (M or) 0.43429 *&c.* is = the Fluxion of the Common Log. of the ſaid Number.

SCHOLIUM.

22. THOUGH hitherto we have ſuppoſed when one variable quantity in a Fluential * Expreſſion increaſes, that, the others, if any, increaſe likewiſe: yet, it often happens, that ſome of them decreaſe while the others increaſe; in which caſe, the Fluxions of the decreaſing are negative with reſpect to thoſe of the increaſing quantities: and therefore, the Signs of the terms affected with them ought to be changed.

Thus, if whilſt x increaſes y decreaſes, the Fluxion of the rectangle xy will be expreſſed by $\dot{x}y - x\dot{y}$.

For, let the flowing or variable rectangle A B be continually increaſing by the parallel motion of the variable line C B moving from the ſituation A F along the line A E; and be continually diminiſhing by the motion of the variable line D B, parallel to the line A E, and approaching towards it along the line F A; that is, let the ſide A C = D B continually increaſe whilſt the Side A D = C B continually decreaſes. Now, putting A C = x, and A D = y, it follows from *Fig.* 8.

---

* By a Fluential Expreſſion is meant that which contains one, two, or more variable quantities.

from *art.* 15. *dem.* 1°. that the Fluxion of the Increaſe will be $=\dot{x}y$, and that the Fluxion of the Decreaſe will be $=x\dot{y}$: therefore, the Fluxion of the Decreaſe ſubtracted from the Fluxion of the Increaſe, leaves the Fluxion of the rectangle xy or the Velocity with which it flows $=\dot{x}y-x\dot{y}$.

IF what has been ſaid be well underſtood, it is hoped, the Learner will meet with few difficulties in the Application thereof; to which we now proceed.

CHAP. III.

Of drawing Tangents to Curves.

23. A *Tangent* is a right line which coincides with a Curve in a point, and there ſhews it's Direction, that is, the Inclination it bears to the Axis or the Angle it makes with the Ordinate.

Fig. 9. 10. Thus, if the right line T B coincides with the Curve A Y at any Point B, the ſaid line is a Tangent to it at that Point: and, becauſe no two lines can coincide unleſs they have the ſame direction, therefore, the direction of the Tangent is properly the direction of the Curve at the point of contact.

Now, in general, what is requiſite in order to draw the Tangent to any point B, is to find the right line C T, called the Subtangent; or, the diſtance of the point T from the ordinate C B through which the Tangent muſt paſs. And to effect this,

24. Let cb be ſuppoſed parallel to the ordinate C B, and B n equal and parallel to C c: then, if the line cb be removed towards C B, in a parallel motion, until it coincides with it; the moment before it's coincidence, the triangle B nb will be in it's evaneſcent State; or, which is the ſame thing, if the ſaid two lines be ſeparated from their coincidence, the very firſt moment of their ſeparation produces the ſaid triangle in it's naſcent ſtate; and in that moment, the line nb, terminated by the line B n at one end and by the curve at the other, comes indefinitely near to touch the Tangent T B produced: conſequently the triangles bnB and B C T come then indefinitely near to ſimilarity, and may be conſidered as ſimilar: wherefore, by 4 E. 6. $bn : n\mathrm{B} :: \mathrm{BC} : \mathrm{CT}$; that is, (putting the abſciſs A C $= x$, the ordinate C B $= y$, C c = B $n = x'$, and $bn = y'$,)

y'

$y' : x' :: y : \frac{x'y}{y'} = CT$; or, (*art.* 7.) $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} = CT$.

Or, Suppoſe the indefinite right line A Z to move with a parallel and uniform motion along the axis A X; and, at the ſame time, a point to move from A along the ſaid line A Z with ſuch velocity as always to be in the curve A Y: then, when the line A Z comes to the ſituation C B, if the point moving along thereon were to continue on with an uniform motion, and the ſame degree of velocity with which it arrives at B, it is evident it would move along the Tangent T B produced; and therefore, when the point A or C arrives at *c*, the point moving along the line A Z would arrive at *m*: and, becauſe velocity is always as the ſpace uniformly deſcribed in a given time, therefore C *c* or B *n* will be as the velocity with which the point A moves along the axis, *n m* as the velocity with which the point moves from B along the line A Z, and B *m* as the velocity with which the point deſcribes the curve at B or moves from B to *m*; that is, C *c* or B *n* will be as the Fluxion of the abſciſs A C, *n m* as the Fluxion of the ordinate C B, and B *m* as the Fluxion of the curve at the point B: but the triangles *m n* B and B C T are ſimilar; therefore, by 4 E. 6. *mn* : *n*B : : BC : C T; that is, the Fluxion of the ordinate : the Fluxion of the abſciſs : : the ordinate : the ſubtangent; or, (putting the abſciſs A C $= x$, and the ordinate C B $= y$,) $\dot{y} : \dot{x} : : y : \frac{\dot{x}y}{\dot{y}} =$ C T; as before.

25. Now, $\frac{\dot{x}y}{\dot{y}}$ is a General Expreſſion for the Subtangent of every curve whoſe abſciſs is x and ordinate y; but, as it is embaraſſed with the Fluxions of x and y, ſo the whole Buſineſs is to exterminate them: and to do this, put the Equation of the curve into Fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ in terms that are all affected with $\dot{y}$, or, of $\dot{y}$ in terms that are all affected with $\dot{x}$; then, if for $\dot{x}$ or $\dot{y}$ we ſubſtitute it's value thus found, in this general expreſſion, *viz.* $\frac{\dot{x}y}{\dot{y}}$, we ſhall have the Subtangent C T in known terms, or free from Fluxions; by which the ſought Tangent T B, to a given point in the curve, may be eaſily drawn.

26.

26. *Note.* When the Fluxions of x and y, or of the abſciſs and ordinate, are Negative to each other, that is, when x increaſes if y decreaſes; or *vice verſa*; then, the General Expreſſion $(\frac{\dot{x}y}{\dot{y}})$ for the Subtangent will be $-\frac{\dot{x}y}{\dot{y}}$; wherein, the Negative ſign only ſignifies, that, the ſubtangent lies on the other ſide of the ordinate with regard to the abſciſs x. But, in finding the Fluxion of the Equation of the curve, the Fluxions of x and y muſt not be conſidered as Negative to each other, if you would have the ſought expreſſion for the ſubtangent come out Affirmative.

EXAMPLE I.

27. To draw a *Tangent* to a *Circle* *.

Fig. 11. Put the radius E A or E D $= a$, abſciſs A C $= x$, and ordinate C B $= y$; then, C D $= 2a - x$. Now, by 35 E. 3. A C × C D $=$ C B^2, that is, $2ax - x^2 = y^2$; and this equation put into Fluxions is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$; which divided by $2a - 2x$, makes

---

* By the word *Circle*, is generally meant the *Space* bounded by a Curve every-where equidiſtant from a fixed point; but ſometimes the *Curve* itſelf. (See Sir *Iſaac Newton's Arithmetica Univerſalis.*)

$\dot{x} = \frac{y\dot{y}}{a-x}$; which ſubſtituted for $\dot{x}$ in the general expreſſion for the Subtangent, (*viz.* $\frac{\dot{x}y}{\dot{y}}$, *art.* 25.) makes the Subtangent CT $= \frac{y^2}{a-x}$ (which, by writing $2ax - x^2$ for it's above value, *viz.* y^2; is) $= \frac{2ax-x^2}{a-x} = \frac{CB^2}{EC}$. Wherefore, if the diſtance ſignified by this expreſſion be ſet off from the point C, in the diameter DA produced, we ſhall have the point T through which the Tangent to the point B muſt paſs.

Conſtruction.

Through the point B deſcribe the ſemicircle EBT: then will T be the point from which the Tangent to the point B is to be drawn. For, by 31 E. 3. the angle EBT will be right; and therefore, by 8 E. 6. the triangles ECB and BCT will be ſimilar, and by 4 E. 6. EC : CB :: BC : CT; $\therefore$ $CT = \frac{CB^2}{EC}$.

EXAMPLE II.

Fig. 12. 28. To draw a *Tangent* to a *Parabola*.

Suppoſe F to be the focus; and PR the parameter, which put $= a$; alſo, put the abſciſs AC $= x$, and ordinate CB $= y$. Now, by a well known property of the curve, $PR \times AC = CB^2$, that is, $ax = y^2$; the Fluxion of which equation is $a\dot{x} = 2y\dot{y}$; therefore, $\dot{x} = \frac{2y\dot{y}}{a}$; which ſubſtituted for $\dot{x}$, makes the general expreſſion for the Subtangent CT (*viz.* $\frac{\dot{x}y}{\dot{y}}$ *art.* 25.) $= \frac{2y^2}{a} =$ (by ſubſtituting ax for y^2 it's value,) $\frac{2ax}{a} = 2x$. So that, the Subtangent CT is double the abſciſs AC; and conſequently, AT is $=$ AC.

EXAMPLE III.

Fig. 13. 29. To draw a *Tangent* to an *Ellipſis*.

Put the tranſverſe diameter AD $= a$, conjugate NO $= b$, abſciſs AC $= x$, and ordinate CB $= y$. Now, by a well known property of the curve, $AD^2 : NO^2 :: AC \times CD : CB^2$; that is, $a^2 : b^2 :: x \times \overline{a - x} :$

$\frac{b^2}{a^2}\times\overline{ax-x^2}=y^2$; the Fluxion of which equation is $\frac{b^2}{a^2}\times\overline{a\dot{x}-2x\dot{x}}=2y\dot{y}$; and this divided by $\frac{b^2}{a^2}\times\overline{a-2x}$, gives $\dot{x}=\frac{2a^2y\dot{y}}{ab^2-2b^2x}$; which ſubſtituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ (the general expreſſion for the Subtangent, *art.* 25.) makes the Subtangent CT $=\frac{2a^2y^2}{ab^2-2b^2x}=$ (by writing $\frac{b^2}{a^2}\times\overline{ax-x^2}$ for y^2 it's equal,) $\frac{2a^3b^2x-2a^2b^2x^2}{a^3b^2-2a^2b^2x}=\frac{2ax-2x^2}{a-2x}$. Whence we may obſerve, that, AT is $\left(=CT-CA=\frac{2ax-2x^2}{a-2x}-x\right)$ $=\frac{ax}{a-2x}=$ that Part of the Subtangent which falls *without* the curve.

Conſtruction.

Make $Cv=CA$; erect the perpendicular Ar, terminated by a right line drawn from E through v; laſtly, make $AT=Ar$: then will T be the point from which the Tangent to the Point B muſt be drawn.

For, by 4 E. 6. E C : $Cv :: EA : Ar$, that is, $\frac{1}{2}a - x : x :: \frac{1}{2}a : \frac{ax}{a-2x} = AT$.

EXAMPLE IV.

Fig. 14. 30. To draw a *Tangent* to an *Hyperbola.*

Put the tranſverſe diameter $DA = a$, conjugate $NO = b$, abſciſs $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve, $DA^2 : NO^2 :: DC \times AC : CB^2$, that is, $a^2 : b^2 :: \overline{a+x} \times x : \frac{b^2}{a^2} \times \overline{ax+x^2} = y^2$; and, by putting both ſides of this equation of the curve into Fluxions, we ſhall have $\frac{b^2}{a^2} \times \overline{a\dot{x} + 2x\dot{x}} = 2y\dot{y}$; therefore, by diviſion, $\dot{x} = \frac{2a^2y\dot{y}}{ab^2 + 2b^2x}$, which multiplied by $\frac{y}{\dot{y}}$, or, which is the ſame, ſubſtituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$, (the general expreſſion for the Subtangent, *art.* 25.) makes the Subtangent $CT = \frac{2a^2y^2}{ab^2 + 2b^2x}$ (which, by writing for y^2 it's equal $\frac{b^2}{a^2} \times \overline{ax + x^2}$

$\overline{ax + x^2}$, is) $= \frac{2ax + 2x^2}{a + 2x}$. So that, AT, that Part of the Subtangent *without* the curve, is $= \frac{2ax + 2x^2}{a + 2x} - x = \frac{ax}{a + 2x}$.

Conſtruction.

Make Cv = CA; erect the perpendicular Ar, terminated by the right line Ev; laſtly, make AT = Ar: then will T be the point from which the Tangent to the point B muſt be drawn. For, by 4E. 6. EC : Cv :: EA : Ar, that is, $\frac{1}{2}a + x : x :: \frac{1}{2}a : \frac{ax}{a + 2x} =$ AT.

SCHOLIUM.

31. FROM the three foregoing Examples, it may be obſerved, that, in the *Parabola* (*fig.* 12.) the internal part of the Subtangent, *viz.* the abſciſs AC, is always *equal* to the external part AT: In the *Ellipſis* (*fig.* 13.) the internal part AC is always *leſs* than the external part AT: And, that, in the *Hyperbola* (*fig.* 14.) the internal part AC is always *greater* than the external part AT.

Ex-

EXAMPLE V.

Fig. 15. 32. To draw a *Tangent* to an *Hyperbola* between it's Aſymptotes; that is, taking one of it's Aſymptotes for an Axis.

Let E Z and E T be the aſymptotes of the hyperbola YAB, whoſe vertex is A; draw AP and BC parallel to the aſymptote EZ; then will A P be the parameter and equal to P E, which put $= a$; E C an abſciſs, which put $= x$; and C B an ordinate, which put $= y$. Now, becauſe when x increaſes y decreaſes; therefore $\dot{x}$ and $\dot{y}$ are negative to each other, and the general expreſſion for the Subtangent is $-\frac{\dot{x}y}{\dot{y}}$; where the negative ſign ſhews, that, the point T lies on the other ſide of the ordinate C B with regard to E, (*art.* 26.). By the well known property of the curve, E C : E P :: PA : CB, that is, $x : a :: a : y$; $\because x = \frac{a^2}{y}$; the Fluxion of which equation is $\dot{x} = \frac{-a^2\dot{y}}{y^2}$; and this value of $\dot{x}$ being ſubſtituted for it in the above general expreſſion for the Subtangent,

tangent, *viz*: $-\frac{\dot{x}y}{\dot{y}}$, makes the Subtangent CT $= \frac{a^2 y\dot{y}}{y^2\dot{y}} = \frac{a^2}{y} =$ (by writing xy for it's equal a^2,) $\frac{xy}{y} = x$. So that, CT muſt be $=$ CE.

EXAMPLE VI.

33. To draw a *Tangent* to the *Conchoid* of *Nicomedes* *. Fig. 16.

Let fall the perpendicular BH on the aſymptote EZ, and draw BC equal and parallel to HE. Put PE $= a$, EA $= b =$ FB, EC $= x =$ HB, and CB $= y =$ EH. Now, by 47 E. 1. $\overline{FB^2 - BH^2}|^{\frac{1}{2}} =$ HF, that is, $\overline{b^2 - x^2}|^{\frac{1}{2}} =$ HF; and, becauſe the triangles PCB and BHF are ſimilar, by 4 E. 6. BH : HF :: PC : CB, that is, $x : \overline{b^2 - x^2}|^{\frac{1}{2}} :: a + x : y$; $\because y = \frac{a+x}{x} \times \overline{b^2 - x^2}|^{\frac{1}{2}}$; which is the equation of the curve; the Fluxion of which is $\dot{y} = \frac{x\dot{x} - .\overline{a+x}\,\dot{x}}{x^2} \times \overline{b^2-x^2}\big|^{\frac{1}{2}} - \frac{1}{2} \times \overline{b^2-x^2}\big|^{-\frac{1}{2}} 2x\dot{x} \times \frac{a+x}{x} = \frac{-a\dot{x}}{\dot{x}^2} \times \overline{b^2-x^2}\big|^{\frac{1}{2}} - \frac{x\dot{x}}{\overline{b^2-x^2}\big|^{\frac{1}{2}}} \times \frac{a+x}{x} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \overline{b^2-x^2}\big|^{\frac{1}{2}}}$; which ſubſtituted for $\dot{y}$, in $-\frac{\dot{x}y}{\dot{y}}$, the general expreſſion for the Subtangent when $\dot{x}$ and $\dot{y}$ are negative to each other, (*art.* 26.) makes the Subtangent $CT = \frac{yx^2 \times \overline{b^2-x^2}\big|^{\frac{1}{2}}}{ab^2+x^3}$ (which, by ſubſtituting for y it's equal in the above equation of the curve, is) $= \frac{\overline{a+x} \times \overline{b^2-x^2} \times x^2}{ab^2x+x^4} = \frac{\overline{a+x} \times \overline{b^2x-x^3}}{ab^2+x^3}$.

---

* This Curve is thus generated.—From a fixed Point P, which is called the Pole of the Conchoid, let a number of right lines PA, PB, PD, &c. be drawn, cutting the right line EZ, which is an aſymptote to the Curve; and let the diſtances EA, FB, GD, &c. be made equal to each other, and a line be drawn through the points A, B, D, &c. then will this line be a curve, called by it's Inventor *Nicomedes* a *Conchoid*.

Conſtruction.

Make $Bn = PC$, draw nr parallel to CP, make $Fv = nr$, draw the right line vC, parallel to which draw BT: then will BT be a Tangent to the point B. For, by 4 E. 6. PC : CB : : PE : EF, that is $a+x : y :: a :$

$\frac{ay}{a+x} =$ EF; and BC : CP :: Bn : nr, that is, $y : \overline{a+x} :: \overline{a+x} : \frac{\overline{a+x}|^2}{y} = nr = \text{F}v$; $\therefore \text{E}v = (\text{EF} + \text{F}v =) \; \frac{ay}{a+x} + \frac{\overline{a+x}|^2}{y}$. Again, vE : EC :: BC : CT, that is, $\frac{ay}{a+x} + \frac{\overline{a+x}|^2}{y} : x :: y : \frac{xy^2 \times \overline{a+x}}{ay^2 + \overline{a+x}|^3} =$ CT. But, by the equation of the curve, $y^2 = \frac{\overline{a+x}|^2}{x^2} \times \overline{b^2 - x^2}$; which ſubſtituted for y^2, makes CT $= \frac{\overline{a+x} \times \overline{b^2x - x^3}}{ab^2 + x^3}$.

EXAMPLE VII.

34. To draw a *Tangent* to the *Ciſſoid* of *Diocles* *. *Fig.* 17.

Let ABD be the Ciſſoid, whoſe generating ſemicircle is AFE, and aſymptote EZ an indefinite right line perpendicular to the diameter AE. Put the diameter AE $= a$, abſciſs AC $= x$, ordinate CB $= y$. Now, becauſe by the generation of the curve, the arches E*b* and A*d* are equal, therefore $bb =$ dC, and bE $=$ CA; and, becauſe the triangles bbA and BCA are alike, therefore, by 4 E. 6. A*b* : *bb* :: AC : CB; but, by 13 E. 6. A*b* : *bb* :: *bb* : *b*E; ∵ *bb* : *b*E :: AC : CB; that is, *d*C : CA :: AC : CB; or (becauſe CE $= a - x$, and by 35 E. 3. C*d* $= \overline{\mathrm{AC} \times \mathrm{CE}}|^{\frac{1}{2}}$ $= \overline{ax - x^2}|^{\frac{1}{2}}$,) $\overline{ax - x^2}|^{\frac{1}{2}} : x :: x : y$; ∵ $x^2 = y \times \overline{ax - x^2}|^{\frac{1}{2}}$; the ſquare of which equation divided by x, is $x^3 = ay^2 - xy^2$; which is the equation of the curve; and the Fluxion of this equation is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$; therefore, by tranſpoſition and diviſion, $\dot{x} = \frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$, which ſubſtituted for $\dot{x}$, makes the Subtangent CT $(= \frac{\dot{x}y}{\dot{y}}$, *art.* 25.) $= \frac{2ay^3 - 2xy^2}{3x^2 + y^2} =$ (by writing for y^2 it's equal $\frac{x^3}{a - x}$,) $\frac{2ax - 2x^2}{3a - 2x}$. Whence we may obſerve, that, AT, the Difference between the abſciſs AC and ſubtangent TC, is $= \frac{ax}{3a - 2x}$.

---

* This Curve is thus generated.—In the ſemicircle AFE, make any two arches E a and A c, or E b and A d, equal to one another; and through the points a, b, d, c, let right lines be drawn perpendicular to the diameter AE, and tranſverſe lines from the point A; then, from A, through the points of interſection ⊙, B, D, &c. draw a line A ⊙ B D &c. and it will be a Curve, called by it's Inventor *Diocles* a *Ciſſoid*.

Conſtruction.

Biſect the radius eE in g, and the abſciſs AC in f; draw the perpendicular Ar equal to fg; make $rn = Af$; and on the point n erect the perpendicular nv, terminated by the right line er; laſtly, draw vT equal and parallel to nA: then will T be the point from which the Tangent to the point B muſt be drawn. For, $rn = Af = \frac{1}{2}x$, and $rA = Ag - Af = \frac{1}{4}a - \frac{1}{2}x$; and by 4 E. 6. $rA : Ae :: rn : nv$ or AT; that is, $\frac{1}{4}a - \frac{1}{2}x : \frac{1}{2}a :: \frac{1}{2}x : \frac{ax}{3a - 2x} = AT$.

EXAMPLE VIII.

35. To draw a *Tangent* to the *Cycloid**. Fig. 18.

Put OA the radius of the generating circle $= a$, abſciſs AC $= x$, ordinate CB $= y$, CG $= s$, and the arch AG $= z$. Now, by the nature of the curve, CB $=$ CG $+$ arch GA; that is, $y = s + z$; (for, when the ſemicircle

---

* This Curve is thus generated.—Let a circle roll along upon a right line until it performs one revolution, that is, until it meaſures out a right line equal to it's circumference; then, that point in the circle which firſt touched the right line will deſcribe the curve called a *Cycloid*; which curve, it is ſuppoſed, was firſt invented by Cardinal *Cuſanus*; whoſe works, in which this figure is found, were tranſcribed by *J. Scoblant* in the year 1451.

AGF, generating the ſemicycloid ABD, is in the poſition BRK, the arch BR or GF muſt evidently be equal to RD, and the arch RK or BM or GA be equal to RF or *t*C; but CG is $=t$B, and therefore C$t=$GB: conſequently, arch GA $=$ GB; and therefore, &c.) and the Fluxion of this equation of the curve is $\dot{y}=\dot{s}+\dot{z}$. Let *cg* be ſuppoſed indefinitely near and parallel to CG, and G*n* equal and parallel to C*c*; that is, let G$g=z'$, $gn=s'$, and nG$=$C$c=x'$: then, ſuppoſing G*g* to be a little right line perpendicular to the radius GO, the angles *g*G*n* and OGC will be equal; for, if to either of them the angle OG*n* be added, the ſum will be a right angle; and therefore, the right angled triangles *gn*G and OCG are alike: conſequently, by 4 E. 6. *gn* : *n*G :: OC : CG; that is, $s' : x' :: a-x : s$; $\because s'=\frac{a-x}{s}x'$, or (*art.* 7.) $\dot{s}=\frac{a-x}{s}\dot{x}$; and, *g*G : G*n* :: OG : GC; that is, $z' : x' :: a : s$; $\because z'=\frac{a}{s}x'$, or, $\dot{z}=\frac{a}{s}\dot{x}$. Now, by ſubſtituting $\frac{a-x}{s}\dot{x}$ for $\dot{s}$, and $\frac{a}{s}\dot{x}$ for $\dot{z}$, in the above Fluxion of the equation of the curve, we have $\dot{y} = \frac{a-x}{s}\dot{x} + \frac{a}{s}\dot{x} = \frac{2a-x}{s}\dot{x}$; and this ſubſtituted for $\dot{y}$, makes the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *art.* 25.) $= \frac{sy}{2a-x}$.

Conſtruction.

Draw the chord or right line AG, and parallel to it draw TB: then will TB be a Tangent to the curve at the point B. For, the triangles FCG and BCT are ſimilar; therefore, by 4 E. 6. FC : CG :: BC : CT, that is, $2a-x : s :: y : \frac{sy}{2a-x} = \text{CT}$.

Or,

Draw the right line EG perpendicular to the radius GO and equal to GB; and through the point E draw the right line TB: then will the ſaid line TB be a Tangent to the curve at the point B. For, let *gb* be ſuppoſed indefinitely near and parallel to GB; then, becauſe the arch AG $=$ GB $= gv$, and the arch AG $+$ Gg $= gv + vb$; therefore Gg or B$v = vb$; conſequently, if we ſuppoſe G*g* to be a little right line perpendicular to the radius OG or coinciding with the right line EG produced, and B*v* to be a little right line parallel to it, and alſo B*b* to be a little right line coinciding with a Tangent to the curve at the point B, the triangles *bv*B and BGE will be ſimilar: therefore, &c.

EXAMPLE IX.

Fig. 19. 36. To draw a *Tangent* to the Curve AB, whoſe Equation (putting GC$=x$, CB$=y$, and $a =$ a given quantity *more* than 1,) is $a^x = y$.

Put A$=$ the Hyperbolic Logarithm of a, and Y$=$ the Hyperbolic Logarithm of y; then, by the nature of Logarithms, $x\mathrm{A} = \mathrm{Y}$; the Fluxion of which equation is $\dot{x}\mathrm{A} = \dot{\mathrm{Y}} =$ (by *art.* 31) $\frac{\dot{y}}{y}$; which divided by A, makes $\dot{x} = \frac{\dot{y}}{\mathrm{A}y}$; and this ſubſtituted for $\dot{x}$, makes the Subtangent CT $(= \frac{\dot{x}y}{\dot{y}}$, *art.* 25.) $= \frac{y}{\mathrm{A}y} = \frac{1}{\mathrm{A}}$. Whence we may obſerve, that, the Subtangent being an invariable quantity, the

Curve AB is the *Logarithmic* Curve, whoſe Aſymptote is GF *.

37. HITHERTO we have treated only of Curves referred to an Axis; or, of thoſe whoſe ordinates are Parallel to one another: We ſhall therefore now proceed to the drawing of Tangents to *Spirals*; or, to thoſe Curves whoſe ordinates iſſue from one and the ſame fixed Point: Where *note*, the ordinate and ſubtangent are always Perpendicular to each other.

38. SUPPOSE Cb indefinitely near to CB; that is, let the ∠ BCb be ſuppoſed indefinitely ſmall; and with the ordinate CB, as a radius, let the indefinitely ſmall arch Bn be deſcribed; which being conſidered as a little right line perpendicular to Cb, and the indefinitely ſmall part of the curve Bb as coinciding with a Tangent to it at the point B; then, becauſe the ∠ BCb is indefinitely ſmall, *Fig.* 20.

---

[*] This Curve is thus generated.—In the indefinite right line GF, make GK=KC=CF, &c. and the perpendiculars GA, KI, CB, FD, &c. in geometrical proportion continued: then, a curve drawn through the points A, I, B, D, &c. will be the *Logarithmic* Curve; which is ſo called, becauſe the diſtances GK, GC, GF, &c. being in arithmetical progreſſion, are as the Logarithms of the ordinates KI, CB, FD, &c.

ſmall, the △s bnB and BCT will be indefinitely near to ſimilarity; that is, in the very firſt moment of the exiſtence of the △bnB, the ſaid △ may be conſidered as ſimilar to the △ BCT; therefore, if we put the ordinate $CB = y$, $Bn = x'$, and $nb = y'$, by 4 E. 6. we ſhall have $y' : x' :: y : CT$; that is, (*art.* 7.) $\dot{y} : \dot{x} :: y : CT$; $\because CT = \frac{\dot{x}y}{\dot{y}}$; which is the ſame General Expreſſion for the Subtangent as that before found for curves referred to an Axis, *art.* 25.

Or, Let the indefinite right line CZ turn, like the radius of a circle, round the fixed point or center C with an uniform motion; and, at the ſame time, let a point moving along thereon, generate, or move with ſuch degrees of velocity as always to be in the Curve CBY, to which ſuppoſe the right line TG a Tangent at the point B; alſo, let B*m* be a Tangent to the circular arch DB*n* deſcribed with the ordinate CB as a radius. Now, when the point generating the curve CY arrives at B, if it was to continue on in the ſame direction, with an uniform motion, and the ſame degree of velocity that it arrives thereat, it would move along the Tangent TB produced, or right line BG; which right line would always be as the Fluxion of the Spiral at the point B: So likewiſe, if we ſuppoſe a point to move from B, in the ſame direction, and with the ſame uniform motion, that the point generating the circular arch DB*n* arrives at B, it would move along the Tangent or right line B*m* perpendicular to the ordinate or radius CB; which right line would always be as the Fluxion of the ſaid arch at the point B: And, ſince the direction of the point moving from C to Z is always perpendicular to that of the point generating the circular arch D*n*; therefore, when the point moving from B to G arrives at G, if G*m* be parallel to ZC, the point moving along B*m* will be arrived at *m*; and therefore, *m*G will be as the Fluxion of the ordinate at the point B. Hence, becauſe the △s G*m*B and BCΓ are ſimilar, by 4 E. 6. G*m* : *m*B : : BC : CT; that is, the Fluxion of the ordinate CB : the Fluxion of the arch DB : : the ordinate CB : the ſubtangent CT; or, (putting the arch DB$=x$, and the ordinate CB$=y$,) $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} =$ CT; as before.

EXAMPLE I.

Fig. 39. 21. To draw a *Tangent* to the *Spiral* of *Archimedes* *.

Put the circumference of the generating circle AFA $=a$, and it's radius CA $=b$; ordinate CB $=y$, arch ARF $=z$; and with the ordinate CB, as a radius, let the circular arch Bn be deſcribed, which put $=x$. Now, by the generation of the curve, $a : b :: z : y$, or $a : b :: \dot{z} : \dot{y}$, $\because \dot{z} = \frac{a\dot{y}}{b}$. But, it is evident, the velocity of the point generating the circle is to the velocity with which the point B generates the arch Bn as CF is to CB; that is, $\dot{z} : \dot{x} :: b : y$; $\because \dot{z} = \frac{b\dot{x}}{y}$. Hence we have $\frac{b\dot{x}}{y} = \frac{a\dot{y}}{b}$; therefore, $b^2\dot{x} = ay\dot{y}$, and

---

* This Curve is thus generated.—With the radius CA, let the circle AFA be deſcribed with an equable motion, or the point A deſcribe equal arches in equal times; and, at the ſame moment of time that the point A begins to generate the circle, let another point begin to move along the ſaid radius from C towards A, and to paſs over it with an uniform motion, and ſuch velocity that it may arrive at A at the very ſame moment of time that the ſaid radius ſhall have deſcribed the circle, or come to be in it's firſt ſituation: then will the point moving along the radius CA, generate, or deſcribe, the curve CBA, called a *Spiral*; the Invention of which is attributed to *Archimedes*.

$\dot{x} = \frac{ay\dot{y}}{b^2}$, which ſubſtituted for $\dot{x}$ in the general expreſſion for the Subtangent, $\frac{\dot{x}y}{\dot{y}}$, (*art.* 38.) makes the Subtangent $CT = \frac{ay^2}{b^2} =$ (becauſe $a : b :: z : y$, or $ay = bz$,) $\frac{byz}{b^2} = \frac{yz}{b}$.

Conſtruction.

With the ordinate CB, as a radius, deſcribe the circular arch BD; and draw the right line CT perpendicular to the radius CF and equal to the arch DB: then will T be the point from which the Tangent to the point B muſt be drawn. For, the ſectors CFRA and CBD are ſimilar; and conſequently, CF : FA :: CB : BD; that is, $b : z :: y : BD = \frac{yz}{b} = CT$.

EXAMPLE II.

40. To draw a *Tangent* to the *Logarithmic Spiral* *. Fig. 22.

Put the curve C*d*B$=z$, and it's correſpondent ordinate CB $= y$. Suppoſe the angles *e*CB and BC*b* to be indefinitely ſmall and equal; and with the ordinate CB, as a radius, deſcribe the circular arch B*n*. Now, by the generation of the curve, C*e* : CB :: CB : C*b*; and therefore, if we conſider the little parts of the curve, *e*B and B*b*, as indefinitely little right lines, by 6 E. 6. the triangles C*e*B and CB*b* will be ſimilar, or the angle made by the curve and ordinate be always the ſame. Conſequently, the Increments of the ordinate and curve are always in an invariable ratio to each other, or as two fixed quantities a and b; that is, $nb : bB :: a : b$; and therefore, if we ſuppoſe B*n* to be a little right line perpendicular to C*b*, by 47 E. 1. $bn : nB :: a : \overline{b^2-a^2}|^{\frac{1}{2}}$; that is, (putting $Bn = x'$, and $nb = y'$,) $y' : x' :: a : \overline{b^2-a^2}|^{\frac{1}{2}}$, or, (*art.* 7.) $\dot{y} : \dot{x} :: a : \overline{b^2-a^2}|^{\frac{1}{2}}$;

---

* This Curve is thus generated.—In the circle AFA, whoſe center is C, let any arches AD, DE, EF, FG, GH, &c. be

---

be made equal to each other; and the right lines C*d*, C*e*, CB, C*b*, C*b*, &c. in geometrical proportion continued: then, a line drawn through the points *β*, *b*, B, *e*, *d*, &c. will be the curve called a *Logarithmic Spiral*; which Name is given to it, becauſe the Arches AD, AE, AF, &c. being in arithmetical progreſſion, are as the Logarithms of the ordinates C*d*, C*e*, CB, &c.

Scholium.

As no geometrical ſeries can, ſtrictly ſpeaking, terminate in o: ſo the *Spiral* B*ed* &c. though it continually tends towards the center C, can never abſolutely arrive thereat; but will approach it within any diſtance that can be aſſigned.

∴ $\dot{x} = \dot{y} \times \frac{\overline{b^2 - a^2}\big|^{\frac{1}{2}}}{a}$; which being ſubſtituted for $\dot{x}$ in the general expreſſion for the Subtangent, viz. $\frac{\dot{x}y}{\dot{y}}$, *art.* 38. makes the ſought Subtangent CT $= y \times \frac{\overline{b^2 - a^2}\big|^{\frac{1}{2}}}{a}$.

Scholium.

We may obſerve, that, (becauſe by 47 E. 1. $TB^2 = BC^2 + CT^2 = y^2 + \frac{b^2 - a^2}{a^2} y^2 = \frac{b^2 y^2}{a^2}$,) the Tangent TB is $= y \times \frac{b}{a}$; and conſequently, the Tangent and Subtangent are to each other as b to $\overline{b^2 - a^2}\big|^{\frac{1}{2}}$. Again, becauſe $nb : bB :: a : b$, that is, (*art.* 7.) $\dot{y} : \dot{z} :: a : b$; therefore, (*art.* 12.) $y : z :: a : b$; or, the ordinate and curve are always in a given or fixed ratio to each other; and therefore $z = y \times \frac{b}{a}$. So that, the Tangent TB and Curve CdB are equal; and ſine ∠BTC : radius :: $a : b$, or, s. ∠BTC $= a$, and rad. (s. ∠BCT) $= b$.

EXAMPLE III*.

Fig. 41. 23, 24. To draw a *Tangent* to the *Spiral* CHBLA, generated by a point moving uniformly along the ſemicircle CDA, from C to A, while the ſaid ſemicircle makes one uniform revolution round the point C as a center.

Let the point *b* be ſuppoſed indefinitely near to B; and with the ordinate CB, as a radius, deſcribe the arch DB*n*. Put the radius of the generating ſemicircle, EB, $= a$, arch CRB $= v$, arch DB $= x$, ordinate CB $= y$, B*b* $= v'$, and *bn* $= y'$. Now, becauſe the circumferences of circles are as their radii; by the generation of the curve $a : 2y :: \dot{v} : \dot{x} = \frac{2y\dot{v}}{a}$; and, becauſe by 20 E. 3. $\angle b\text{EB} = 2\angle \text{BC}b$, $a : y :: v' : 2\text{B}n$, $\therefore \text{B}n = \frac{yv'}{2a}$; but, by 47 E. 1. (ſuppoſing B*b* to be a little right line, and B*n* a little right line perpendicular to C*b*,) $\text{B}b^2 - bn^2 = n\text{B}^2$, that is, $v'^2 - y'^2 = \frac{y^2v'^2}{4a^2}$; from which equation we have $v' = \frac{2ay'}{\overline{4a^2 - y^2}\,|^{\frac{1}{2}}}$, that is, (*art.*

7.)

---

* Invented *Anno* 1756.

7.) $\dot{v} = \frac{2a\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$. Hence, $\dot{x} = \frac{2y}{a} \times \frac{2a\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}} = \frac{4y\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$; which ſubſtituted for $\dot{x}$, makes $\frac{\dot{x}y}{\dot{y}}$, the general expreſſion for the Subtangent CT (*art.* 38.) $= \frac{y}{\dot{y}} \times \frac{4y\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$

$$= \frac{4y^2}{\overline{4a^2-y^2}|^{\frac{1}{2}}}.$$

Corollary.

Since $\dot{x} = \frac{2y\dot{v}}{a}$, that is, $x' = \frac{2yv'}{a}$; and Bn $= \frac{yv'}{2a}$: therefore, $x' = 4$Bn.

Conſtruction.

Draw the chord BG, which biſect in F; and to the point F draw the right line CF; produce the chord or ordinate CB to I, making IB = BC; alſo, draw the right line IT, making the angle CIT = the angle CFB: then will T be the point from which the Tangent to the point B muſt be drawn. For, by 31 E. 3. the angle CBG is right, and therefore the right angled triangles FBC and ICT are ſimilar; wherefore, by 4 E. 6. BF :

BC :: IC : CT; that is, (becauſe by 47 E. 1. $BG = \overline{GC^2 - CB^2}|^{\frac{1}{2}} = \overline{4a^2 - y^2}|^{\frac{1}{2}}$, and therefore $BF = \frac{1}{2} \cdot \overline{4a^2 - y^2}|^{\frac{1}{2}}$,) $\frac{1}{2} \cdot \overline{4a^2 - y^2}|^{\frac{1}{2}} : y :: 2y : \frac{4y^2}{\overline{4a^2 - y^2}|^{\frac{1}{2}}} =$ CT.

CHAP. IV.

Of finding the Maxima *and* Minima, *or* Greateſt *and* Leaſt *of variable Quantities.*

42. A *Maximum* is the greateſt magnitude of a variable Quantity, which naturally increaſes before it arrives at that magnitude and as naturally decreaſes afterwards; and a *Minimum* is the leaſt magnitude, of a variable Quantity, which naturally decreaſes before it arrives at that magnitude and afterwards as naturally increaſes. Therefore, at that certain term where a Quantity becomes a Maximum or a Minimum it can neither be increaſing nor decreaſing; and conſequently, it's Fluxion is = 0.

Fig. 25. Thus, in the triangle AEF, let the equal and parallel ſides AC and DB of the inſcribed and flowing rectangle CD be continually increaſing while the other equal and parallel ſides AD and CB are continually decreaſing;

that is, let the ſaid rectangle be increaſed by the motion of the variable and decreaſing ſide CB, and decreaſed by the motion of the variable and increaſing ſide DB. Then, it is evident, that, while the rectangle is increaſing faſter, or with a greater degree of velocity, by the motion of the line CB, than it is decreaſing by the motion of the line DB, it will be continually Increaſing; and that, when it is decreaſing faſter, or with a greater degree of velocity, by the motion of the line DB, than it is increaſing by the motion of the line CB, it will be continually Decreaſing: therefore, when theſe two degrees of velocity become equal, the rectangle will be a Maximum; and be neither increaſing nor decreaſing; or, the Velocity or Fluxion with which it is decreaſing ſubtracted from the Velocity or Fluxion with which it is increaſing, leaves the Velocity or Fluxion with which it flows $=0$. And, it being manifeſt, that, when the rectangle CD becomes a Maximum, the ſum of the triangles FDB and BCE will be a Minimum; therefore, when the ſum of the ſaid triangles is a Minimum, the Fluxion of the increaſing triangle FDB will be equal to the Fluxion of the decreaſing triangle ECB; and therefore, the Fluxion of the latter ſubtracted from that of the former, gives the Fluxion of the Minimum $= 0$.

43. WHEN any Quantity is a Maximum or a Minimum, all the Affirmative Powers of it will be ſo too; as will alſo, the Sum, Remainder, Product or Quotient, ariſing from it's being added to, ſubtracted from, multiplied or divided by, any invariable or given quantity. But, any Negative * Power of a Maximum will be a Minimum; and any Negative Power of a Minimum will be a Maximum.

Thus, (1°.) if $a-b+\frac{c}{d}\times\overline{ax-x^2}|^{\frac{1}{2}}$ be a Maximum, then will $ax-x^2$ be alſo a Maximum. And, (2°.) if $\overline{\frac{a^2c^2}{x^2}+a^2+c^2+x^2}|^{-1}$ be a Maximum, then will $\frac{a^2c^2}{x^2}+a^2+c^2+x^2$ or $\frac{a^2c^2}{x^2}+x^2$ be a Minimum.

For (1°.) it is evident, that, the greater $ax-x^2$ is, the greater will be $a-b+\frac{c}{d}\times\overline{ax-x^2}|^{\frac{1}{2}}$; and

---

* By an affirmative power, is meant that whoſe Index is affirmative; and, by a negative power, that whoſe Index is negative.

and therefore, when One of theſe expreſſions is a Maximum, the Other will be a Maximum alſo. And (2°.) ſince (*art.* 13.) $\overline{\frac{a^2c^2}{x^2}+a^2+c^2+x^2}\Big|^{-1}$ is $=\frac{1}{\frac{a^2c^2}{x^2}+a^2+c^2+x^2}$, it is plain, that, the leſs the denominator of this fractional expreſſion is, the greater will be the quotient; but the ſaid denominator is evidently the leaſt poſſible when $\frac{a^2c^2}{x^2}+x^2$ is a Minimum.—Therefore, &c.

44. WHEN in the expreſſion for the Fluxion of a Maximum or a Minimum there are two or more *fluxionary* letters, and each is contained in both affirmative and negative terms; then, the ſum of the terms affected with either of them will be $=0$.

Thus, if the Fluxion of a Minimum be $\frac{1}{2}a\dot{x}-\dot{x}y+x\dot{y}-\frac{1}{2}b\dot{y}=0$; then $\frac{1}{2}a\dot{x}-\dot{x}y=0$, and $x\dot{y}-\frac{1}{2}b\dot{y}=0$.

For, let the variable rectangle CD, whoſe diagonal is AB, flow in the triangle EAF. Put FA $=a$, AE $=b$, AC or DB $=x$, and CB or AD $=y$. Then will the triangle FAB be $=\frac{1}{2}ax$, the triangle AEB be $=\frac{1}{2}by$, and the rectangle CD be $=xy$; and therefore, the ſum of the triangles FDB and BCE will be $=\frac{1}{2}ax-xy+\frac{1}{2}by$; which, when the rectangle CD or xy is a Maximum, is evidently a Minimum. Now, it is plain, if x increaſes y muſt decreaſe, and therefore, (*art.* 22.) their Fluxions are negative to each other, and the Fluxion of the ſaid Minimum is $\frac{1}{2}a\dot{x}-\dot{x}y+x\dot{y}-\frac{1}{2}b\dot{y}=0$, and the Fluxion of the ſaid Maximum is $\dot{x}y-x\dot{y}=0$, or, the Fluxion of the triangle ADB or ACB is $\frac{1}{2}\dot{x}y-\frac{1}{2}x\dot{y}=0$: but, the Fluxions of the triangles FAB and BAE are manifeſtly equal, ſince one increaſes as faſt as the other decreaſes. Hence, therefore, (the Fluxion of the triangle ACB being $=0$,) the Fluxion of the triangle FAB is $=$ the Fluxion of the triangle ECB; that is, $\frac{1}{2}a\dot{x}=\dot{x}y$, (*art.* 15. *demon.* 1^{c}.) conſequently, $\frac{1}{2}a\dot{x}-\dot{x}y=0$; and therefore $x\dot{y}-\frac{1}{2}b\dot{y}=0$.

45. IN plane Figures, it is, in effect, the ſame thing to find the *greateſt* Area that can be contained under a *given* Perimeter, as to find a *given* Area under the *leaſt* Perimeter. —It may alſo be obſerved, that, to find the *greateſt* Solid that can be contained under a *given* Surface, is the ſame as to find a *given* Solid under the *leaſt* Surface.

46. GENERALLY, when a variable Quantity admits of a Maximum, it's Minimum is

Nothing; and, when it admits of a Minimum, it's Maximum is Infinite.

EXAMPLE I.

47. To find the point C in the given right line AB, where the rectangle of the parts AC and CB is a *Maximum*, or greater than any other rectangle A*n* × *n*B. Fig. 26.

Put AB $= a$, and AC $= x$; then CB $= a - x$, and therefore the rectangle AC × CB $= x \times \overline{a-x} = ax - x^2 =$ a Maximum. Now, the Fluxion of a Maximum being $= 0$, the Fluxion of $ax - x^2$ muſt be $= 0$; that is, $a\dot{x} - 2x\dot{x} = 0$; which divided by $\dot{x}$, makes $a - 2x = 0$; therefore, $a = 2x$, and $x = \frac{1}{2}a$. Conſequently, the rectangle of the parts AC and CB is a Maximum, or the greateſt poſſible, when the ſaid parts are equal.

Or, Put AC $= x$, and CB $= y$; then AC × CB $= xy =$ a Maximum. Now, it is evident, that, if x increaſes y muſt decreaſe; and therefore the Fluxions of x and y are negative to each other; and conſequently, when xy is a Maximum, the Fluxion of it will be $\dot{x}y - x\dot{y} = 0$: but, it is likewiſe evident, that, the Increment of x is equal to the Decrement of y; or, that the Velocity with which x increaſes is equal to the Velocity with which y decreaſes; that is, $\dot{x}=\dot{y}$: therefore, by dividing by $\dot{x}$ or $\dot{y}$, or by ſtriking both $\dot{x}$ and $\dot{y}$ out, in the above Fluxion of the Maximum xy, we ſhall have $y-x=0$; and therefore $x=y$; as before.

Or, In the variable rectangle Ab, let the ſide Cb, which is equal and parallel to the ſide AD, always be equal to CB: then, while the rectangle is increaſing by the motion of the decreaſing ſide Cb moving from A towards B, it will be decreaſing by the motion of the increaſing ſide bD moving from F towards A. Now, ſince the Velocities of the points C and D, or the Fluxions of the ſides AC and AD, are always equal, (as they evidently muſt be, becauſe the ſum of theſe ſides is always the ſame;) therefore, as long as the ſide AC continues leſs than the ſide AD or Cb, the rectangle will be continually increaſing; and after the ſide AC becomes equal to the ſide Cb, it will be continually decreaſing: therefore, when the rectangle is a Maximum, or when it is neither increaſing nor decreaſing, the ſides AC and Cb are equal to each other, or AC=Cb=CB; as before.

Or, Deſcribe the ſemicircle AbB, and let fall the perpendicular bC. Now, by 35 E. 3.

$AC \times CB = Cb^2$; and therefore $AC \times CB$ is a Maximum when bC is the radius of the ſemicircle, or when the point C biſects AB; as before.

EXAMPLE II.

48. To find the point C in the given right line AB, where $AC^m \times CB^n$ is a *Maximum.*

Put $AB = a$, and $AC = x$; then $CB = a - x$, and $AC^m \times CB^n = x^m \times \overline{a-x}|^n = \overline{x^{\frac{m}{n}} \times \overline{a-x}}|^n =$ a Maximum; therefore, (*art.* 43.) $x^{\frac{m}{n}} \times \overline{a-x} =$ a Maximum; that is, (putting $\frac{m}{n} = e$,) $x^e \times \overline{a-x} = ax^e - x^{e+1} =$ a Maximum; the Fluxion of which is $= 0$; that is, $aex^{e-1}\dot{x} - \overline{e+1}\,x^e\dot{x} = 0$. Now, by dividing this equation by $x^{e-1}\dot{x}$, we have $ae - \overline{e+1}\,x = 0$; therefore, $ae = \overline{e+1}\,x$, and $x = \frac{ae}{e+1}$; that is, by reſtitution; or writing $\frac{m}{n}$ for e, $x = \frac{m}{m+n}a$: whence the point C is determined.

Or, Put $AC = x$, and $CB = y$; then $x^m \times y^n =$ a Maximum; the Fluxion of which is $= 0$; that is, (becauſe when x increaſes y de-

decreaſes, or the Fluxions of x^m and y^n are negative to each other,) $mx^{m-1}\dot{x}\times y^n - ny^{n-1}\dot{y}\times x^m = 0$: but, the Fluxions of x and y are evidently equal; therefore, throw both $\dot{x}$ and $\dot{y}$ out; then it will be $mx^{m-1}\times y^n - ny^{n-1}\times x^m = 0$. Now, by dividing this equation by $x^{m-1}y^{n-1}$, we have $my - nx = 0$; therefore, $nx = my$, and $m : n :: x : y$. So that, the ſegments AC and CB are in direct proportion to the Indices of their Powers; and $x = \frac{m}{n}y = \frac{m}{n}\times\overline{a-x} = \frac{ma-mx}{n}$; therefore, $nx = ma - mx$, and $x = \frac{m}{m+n}a$; as before.

EXAMPLE III.

49. To find the *greateſt* Cone that can be inſcribed in a given Sphere.

Fig. 27. Put AD, the diameter of the Sphere, $= a$; .78539 &c. (the area of a circle whoſe diameter is 1,) $= c$; and AC, the altitude of the cone, $= x$: then $CD = a - x$. Now, by 35 E. 3. $AC\times CD = CB^2$; that is, $x\times\overline{a-x} = ax - x^2 = CB^2$; therefore, (becauſe the ſquare of the diameter is 4 times the ſquare of the radius,) by 2 E. 12. $4acx - 4cx^2 =$ the area of the cone's baſe; which, by

10 E. 12, drawn into $\frac{1}{3}x$, is $\frac{4}{3}acx^2 - \frac{4}{3}cx^3$ $=$ the cone's ſolidity, a Maximum: therefore, by *art.* 43. $ax^2 - x^3 =$ a Maximum; the Fluxion of which is $= 0$; that is, $2ax\dot{x} - 3x^2\dot{x} = 0$; which divided by $x\dot{x}$, gives $2a - 3x = 0$; therefore, $3x = 2a$, and $x = \frac{2}{3}a$. So that, the cone will be a Maximum, or the greateſt poſſible, when it's altitude is $=$ two-thirds of the ſphere's diameter.

EXAMPLE IV.

50. To find the internal dimenſions of a cylindrical Cup, whoſe capacity is given $= a$, when the ſaid Cup is made with the *leaſt* poſſible quantity of Silver of a given thickneſs.

Put the diameter $= x$, and .78539, &c. (the area of a circle whoſe diameter is 1,) $= c$; then, by 2 E. 12. $cx^2 =$ the area of the bottom, and therefore $\frac{a}{cx^2} =$ the altitude: but, $4cx =$ the circumference of the bottom, and therefore $4cx \times \frac{a}{cx^2} = \frac{4a}{x} =$ the inſide curve ſuperficies. Hence $cx^2 + \frac{4a}{x} =$ the whole inſide ſuperficies; which, becauſe the quantity of ſilver is the Leaſt poſſible, is a Minimum; and therefore it's Fluxion is $=0$; that is, $2cx\dot{x} - \frac{4a\dot{x}}{x^2} = 0$; which multiplied by x^2, is $2cx^3\dot{x} - 4a\dot{x} = 0$; and this divided by $2\dot{x}$, is $cx^3 - 2a = 0$; therefore, $cx^3 = 2a$, and $x = \overline{\frac{2a}{c}}\Big|^{\frac{1}{3}} =$ the diameter; which ſubſtituted for x, makes the above $\frac{a}{cx^2} = \frac{a}{c\times\overline{\frac{2a}{c}}\Big|^{\frac{2}{3}}}$

$$= \frac{a\times\overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}}{c\times\frac{2a}{c}} = \tfrac{1}{2}\times\overline{\frac{2a}{c}}\Big|^{\frac{1}{3}} = \text{the altitude.}$$

So that the diameter is to the altitude as 2 to 1.

Or, Put the diameter $= x$, altitude $= y$, and .78539 &c. $= c$; then, the whole inſide ſuperficies will be $= cx^2 + 4cxy =$ a Minimum; therefore (*art.* 43.) $x^2 + 4xy =$ a Minimum; the Fluxion of which is $=0$; that is, (ſuppoſing x to increaſe and y to decreaſe, or the Fluxions of x and y to be negative to each other, as they evidently muſt be,) $2x\dot{x} + 4\dot{x}y - 4x\dot{y} = 0$. But, $cx^2y = a$, and conſequently $x^2y = \frac{a}{c}$; the Fluxion of which equation,

tion, (becauſe the Fluxion of $\frac{a}{c}$ is $=0$; and the Fluxion of y is negative,) is $2x\dot{x}y-x^2\dot{y}=0$; therefore $x^2\dot{y}=2x\dot{x}y$; which equation divided by x^2, makes $\dot{y}=\frac{2\dot{x}y}{x}$. Hence, this value of $\dot{y}$ being ſubſtituted for it in the above Fluxion of the Minimum, we have $2x\dot{x}+4\dot{x}y-8\dot{x}y=0$; that is, $2x\dot{x}-4\dot{x}y=0$; which equation divided by $2\dot{x}$, gives $x-2y=0$; conſequently $x=2y$; as before.

EXAMPLE V.

51. To find the internal dimenſions of a Ciſtern in the form of a rectangular ſolid, (that is, whoſe bottom and all four ſides are rectangular,) when it's capacity is $=a$, and it is made with the *leaſt* poſſible quantity of Lead of a given thickneſs.

Put the inſide length AC or DB $=x$, *Fig.* 28. breadth AD or CB $=y$, and depth BF or CE $=z$; then, $xyz=a$, and therefore, $x=\frac{a}{yz}$. Now, the inſide ſuperficies of the bottom and four ſides is $=AC\times CB+2AC\times CE+2CB\times BF=xy+2xz+2yz=$ (by ſubſtituting $\frac{a}{yz}$ for x,) $\frac{a}{z}+\frac{2a}{y}+2yz$; which, becauſe the ciſtern is made with the Leaſt poſſible quantity of lead, is a Minimum; and therefore it's Fluxion is $=0$; that is, $-\frac{a\dot{z}}{z^2} - \frac{2a\dot{y}}{y^2} + 2\dot{y}z + 2y\dot{z} = 0$. Now, (*art.* 44.) the ſum of the terms affected with $\dot{y}$ is $=0$, and the ſum of the terms affected with $\dot{z}$ is $=0$; that is, $-\frac{2a\dot{y}}{y^2} + 2\dot{y}z = 0$, and $-\frac{a\dot{z}}{z^2} + 2y\dot{z} = 0$; the former of which equations multiplied by y^2, gives $-2a\dot{y} + 2y^2\dot{y}z = 0$, and this divided by $2\dot{y}$, is $-a + y^2z = 0$, $\because y^2z = a = xyz$, therefore, $y = x$: and the latter of the ſaid equations multiplied by z^2, gives $-a\dot{z} + 2yz^2\dot{z} = 0$; which divided by $\dot{z}$, is $-a + 2yz^2 = 0$, $\because 2yz^2 = a = xyz$, therefore, $2z = x$. Hence, $x = y = 2z$; that is, the length and breadth will be equal, and each equal to twice the depth.

Or, The length, breadth, and depth, being x, y, and z, reſpectively; and the inſide ſuperficies $= xy + 2xz + 2yz =$ a Minimum, as before; if we ſuppoſe x and y to increaſe, then z muſt neceſſarily decreaſe; that is, the Fluxions of x and y being affirmative, the Fluxion of z will be negative; and therefore, the Fluxion of the Minimum will be

$\dot{x}y+x\dot{y}+2\dot{x}z-2x\dot{z}+2\dot{y}z-2y\dot{z}=0$. But, $xyz=a$, the Fluxion of which equation, (the Fluxion of a being $=0$, and the Fluxion of z being negative,) is $\dot{x}yz+x\dot{y}z-xy\dot{z}=0$; $\because$ $\dot{z}=\frac{\dot{x}yz+x\dot{y}z}{xy}=\frac{\dot{x}z}{x}+\frac{\dot{y}z}{y}$, which ſubſtituted for $\dot{z}$, in the Fluxion of the Minimum, makes it $\dot{x}y+x\dot{y}-\frac{2x\dot{y}z}{y}-\frac{2\dot{x}yz}{x}=0$; therefore (*art.* 44.) $\dot{x}y-\frac{2\dot{x}yz}{x}=0$, and $x\dot{y}-\frac{2x\dot{y}z}{y}=0$. Now, the firſt of theſe two equations multiplied by $\frac{x}{\dot{x}y}$, is $x-2z=0$, $\because$ $x=2z$: and the latter multiplied by $\frac{y}{x\dot{y}}$, is $y-2z=0$, $\because$ $y=2z$. So that, $x=y=2z$; as before.

Scholium.

In this laſt method of Solution, either of the quantities x, y, or z, might have been ſuppoſed to decreaſe while the others increaſe; or, but one of them to increaſe while the others decreaſe: for, in either caſe, the concluſions would come out exactly the ſame.

Ex-

EXAMPLE VI.

52. A Gentleman wants to ride from the City C to the City D, the Cities being a miles apart. Now, from C to A, which is b miles, or from C to any place B in the road DA, which is perpendicular to the road AC, he can ride after the rate of c miles an hour; but, from A to D he can ride faſter, or after the rate of d miles an hour. To find B, the place to which he muſt directly ride, in order to perform his Journey in the *leaſt* poſſible Time.

Fig. 29.

By 47 E. 1. AD $= \overline{DC^2 - CA^2}|^{\frac{1}{2}} = \overline{a^2 - b^2}|^{\frac{1}{2}}$, which put $= e$; and let AB $= x$; then BD $= e - x$; and therefore, $\frac{e-x}{d} =$ the number of hours the Gentleman will be riding from B to D. Again, by 47 E. 1. CB $= \overline{BA^2 + AC^2}|^{\frac{1}{2}} = \overline{x^2 + b^2}|^{\frac{1}{2}}$; and therefore, $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c} =$ the number of hours he will be riding from C to B. Hence we have $\frac{\overline{x^2+b^2}|^{\frac{1}{2}}}{c} + \frac{e-x}{d} =$ the number of hours he will be performing his journey, which muſt be a Minimum: therefore (*art.*43.) $d\times\overline{x^2+b^2}|^{\frac{1}{2}} - cx$ = a Minimum; the Fluxion of which is $= 0$; that is, $\frac{1}{2}d\times\overline{x^2+b^2}|^{-\frac{1}{2}}\times 2x\dot{x} - c\dot{x} = \frac{dx\dot{x}}{\overline{x^2+b^2}|^{\frac{1}{2}}} - c\dot{x} = 0$; which multiplied by $\overline{x^2+b^2}|^{\frac{1}{2}}$, gives $dx\dot{x} - c\dot{x}\times\overline{x^2+b^2}|^{\frac{1}{2}} = 0$; and this tranſpoſed and divided by $\dot{x}$, is $dx = c\times\overline{x^2+b^2}|^{\frac{1}{2}}$; the ſquare of which is $d^2x^2 = c^2x^2+b^2c^2$; therefore $x^2 = \frac{b^2c^2}{d^2-c^2}$, and $x = \frac{bc}{\overline{d^2-c^2}|^{\frac{1}{2}}}$ = AB.

Or, Let C*b* be ſuppoſed indefinitely near to CB, and the little circular arch B*n* be deſcribed with the radius CB; that is, let B*b* expreſs the naſcent or indefinitely ſmall Increment of AB, and *nb* that of CB. Now, it is evident, if B be the place to which he muſt directly ride, that, the Increments B*b* and *nb* muſt be to each other as *d* to *c*, ſince then only can they be paſſed over in the ſame time: but, the little triangle B*nb* is (or may be taken as) ſimilar to the right angled triangle CAB; (for, the arch B*n*, being indefinitely ſmall, may be conſidered as a little right line perpendicular to Cb; ſo likewiſe, the angle BCb being indefinitely little, the angle CbA or nbB may be conſidered as equal to the angle CBA; and therefore the angle bBn as equal to the angle BCA;) conſequently, by 4 E. 6. $d : c :: CB : BA$; but, when the hypothenuſe and perpendicular are d and c, the baſe, by 47 E. 1. will be $\overline{d^2-c^2}|^{\frac{1}{2}}$. Hence, therefore, $\overline{d^2-c^2}|^{\frac{1}{2}} : (CA)\ b :: c : \frac{bc}{\overline{d^2-c^2}|^{\frac{1}{2}}} = AB$; as before.

Conſtruction.

Through C draw mv parallel to AD, making $Cv = d + c$, and $Cm = d - c$; deſcribe the ſemicircle mkv; on the interſecting point k erect the perpendicular $kr = c$; and through r draw the right line CB: then will B be the point required. For, by 35 E. 3. $\overline{vC \times Cm}|^{\frac{1}{2}} = Ck$, that is, $\overline{d^2-c^2}|^{\frac{1}{2}} = Ck$; and, by 4 E. 6. $Ck : kr :: CA : AB$, that is, $\overline{d^2-c^2}|^{\frac{1}{2}} : c :: b : \frac{bc}{\overline{d^2-c^2}|^{\frac{1}{2}}} = AB$.

EXAMPLE VII.

53. Let the triangle BAD have one angle A in the right line CE: To find a *Maximum* of the Sum of the perpendiculars BC and DE let fall from the other two angles on the right line aforeſaid. *Fig.* 30.

Note, AB $=a$, AD $=b$, and the ∠BAD $=90^\circ$.

Put AC $=x$; then, by 47 E. 1. CB $=\overline{BA^2-AC^2}|^{\frac{1}{2}}=\overline{a^2-x^2}|^{\frac{1}{2}}$. Now, becauſe the ∠BAD is right, the ∠DAE is the complement of the ∠BAC, and is therefore $=$ ∠ABC; conſequently the triangles BAC and ADE are ſimilar; and therefore, by 4 E. 6. BA : AC :: AD : DE, that is, $a : x :: b : \frac{bx}{a}=$ DE. Hence we have BC$+$DE $=$ $\overline{a^2-x^2}|^{\frac{1}{2}}+\frac{bx}{a}=$ a Maximum; the Fluxion of which is $=$ o; that is, $\frac{1}{2}\times\overline{a^2-x^2}|^{-\frac{1}{2}}\times-2x\dot{x}+\frac{b\dot{x}}{a}=$ o, that is, $-\frac{x\dot{x}}{\overline{a^2-x^2}|^{\frac{1}{2}}}+\frac{b\dot{x}}{a}=$ o; which multiplied by $\overline{a^2-x^2}|^{\frac{1}{2}}\times a$, gives $-ax\dot{x}+\overline{a^2-x^2}|^{\frac{1}{2}}\,b\dot{x}=$ o; therefore, by tranſpoſition and dividing by $\dot{x}$, we have $ax=\overline{a^2-x^2}|^{\frac{1}{2}}b$; by involution, $a^2x^2=a^2b^2-b^2x^2$; conſequently, $a^2x^2+b^2x^2=a^2b^2$, and $x^2=\frac{a^2b^2}{a^2+b^2}$; therefore, $x=\frac{ab}{\overline{a^2+b^2}|^{\frac{1}{2}}}$; which ſubſtituted for x, makes the Maximum $BC+DE$ $(=\overline{a^2-x^2}|^{\frac{1}{2}}+\frac{bx}{a})=\overline{a^2+b^2}|^{\frac{1}{2}}$ $=\overline{BA^2+AD^2}|^{\frac{1}{2}}$, that is, by 47 E. 1. $BC+DE=$ the hypothenuſe BD.

Or, Produce BA to *b*, making *b*A = AB; draw the right line *b*D; and perpendicular to CE draw *bc*: then will the triangles A*bc* and ABC be equal and ſimilar, for *bc* is equal and parallel to BC. Now, it is evident, that, the line *b*D, in every ſituation, will be greater than the ſum of the perpendiculars *bc* and DE, excepting when it is perpendicular to the line CE; and that then, the ſaid perpendiculars will coincide with, and their ſum be equal to it. Therefore, the Maximum of the Sum of the perpendiculars *bc* and DE, or of BC and DE, is equal to the ſide *b*D of the triangle A*b*D; which, when the $\angle b$AD or BAD is right, is equal to BD; as before.

EXAMPLE VIII.

54. To find the *greateſt* right angled Triangle ACB that can be inſcribed in the given Quadrant AEF; the right angle being formed by the ſine and coſine BC and CA. *Fig.* 31.

Put the radius or hypothenuſe AB$=a$, and AC$=x$; then, by 47 E. 1. CB$=\overline{a^2-x^2}|^{\frac{1}{2}}$; therefore, the area of the triangle $=\frac{1}{2}x\times\overline{a^2-x^2}|^{\frac{1}{2}}=\overline{\frac{1}{4}a^2x^2-\frac{1}{4}x^4}|^{\frac{1}{2}}=$ a Maximum: conſequently, (*art.* 43.) $a^2x^2-x^4=$ a Maximum; the Fluxion of which is $=0$; that is, $2a^2x\dot{x}-4x^3\dot{x}=0$; which divided by $2x\dot{x}$, makes $a^2-2x^2=0$; therefore $x^2=\frac{1}{2}a^2$; conſequently AC=CB, and the point B biſects the quadrantal arch EF.

Or, Put AB$=a$, AC$=x$, CB$=y$, EB$=z$; and ſuppoſe the point *b* indefinitely near to B, *bc* parallel to BC, and B*n* equal and parallel to C*c*; that is, let B*n*$=x'$, and B*b*$=z'$. Now, if we conſider the Increment B*b* as a little right line perpendicular to the radius AB, the angles *n*B*b* and CBA will be equal, and conſequently the indefinitely little right angled triangle B*nb* will be ſimilar to the right angled triangle BCA; therefore, by

4 E. 6. AB ; BC :: bB : Bn, that is, $a : y :: z' : x'$, or (*art.* 7.) $a : y :: \dot{z} : \dot{x}$, $\therefore \dot{z} = \frac{a\dot{x}}{y}$. But, when the triangle ACB is a Maximum, it is plain, that, AB × $\frac{1}{2}\dot{z}$ is = BC × $\dot{x}$; that is, $\frac{1}{2}a\dot{z} = y\dot{x}$, $\therefore \dot{z} = \frac{2y\dot{x}}{a} =$ (by the above,) $\frac{a\dot{x}}{y}$; whence, $2y^2\dot{x} = a^2\dot{x}$, and $2y^2 = a^2 = x^2 + y^2$. Therefore, $x = y$, or AC = CB; as before.

Or, Suppoſe CD to be always perpendicular to AB. Now, it is evident, the Triangle will be the *greateſt* when CD is a Maximum, that is, when it biſects AB, or is the radius of the circumſcribing ſemicircle ACB; and then, it is plain, AC = CB; as before.

EXAMPLE IX.

55. In the right line AE, which is perpendicular to the indefinite right line AZ, given AC $= a$, and CE $= b$: To find the point B, where the Angle CBE is a *Maximum*. *Fig.* 32.

Put AE $= a + b = c$, and AB $= x$; then, by 47 E. 1, CB $= \overline{a^2 + x^2}|^{\frac{1}{2}}$, and EB $= \overline{c^2 + x^2}|^{\frac{1}{2}}$.

Now, by Trigonometry, CB : radius :: AB : s. ∠ACB, that is, (putting radius = 1,) $\overline{a^2+x^2}\big|^{\frac{1}{2}} : 1 :: x : \frac{x}{\overline{a^2+x^2}\big|^{\frac{1}{2}}}$ = s. ∠ ACB or ECB; and EB : s.∠ECB :: EC : s.∠CBE, that is, $\overline{c^2+x^2}\big|^{\frac{1}{2}} : \frac{x}{\overline{a^2+x^2}\big|^{\frac{1}{2}}} :: b : \frac{bx}{\overline{a^2+x^2}\big|^{\frac{1}{2}}\times\overline{c^2+x^2}\big|^{\frac{1}{2}}}$ = s.∠CBE = a Maximum; therefore (*art.* 43.) $\frac{x^2}{\overline{a^2+x^2}\times\overline{c^2+x^2}} = \frac{1}{a^2c^2x^{-2}+a^2+c^2+x^2}$ = a Maximum; and $a^2c^2x^{-2}+x^2$ = a Minimum; the Fluxion of which is $= 0$; that is, $-2a^2c^2x^{-3}\dot{x}+2x\dot{x}=0$; and this equation divided by $2x\dot{x}$, makes $-a^2c^2x^{-4}+1=0$; therefore $1=a^2c^2x^{-4}=\frac{a^2c^2}{x^4}$, and $x^4=a^2c^2$, and $x=\overline{ac}\big|^{\frac{1}{2}}$. So that AB is a geometrical mean between the diſtances AC and AE; and therefore, by 6 E. 6. the triangles CAB and ~~C~~ABE are ſimilar.

Or *, (in *any* poſition of the line AZ,)— It is evident, that, as long as the angle BCE decreaſes faſter than the angle BEC increaſes, the angle CBE will be increaſing; and that, when the angle BEC increaſes faſter Fig. 32. 33. 34.

---

* Invented *Anno* 1760.

faſter than the angle BCE decreaſes, the angle CBE will be decreaſing: therefore, when the ſaid angle CBE is a Maximum, or when it neither increaſes nor decreaſes, the Fluxions of the angles at C and E will be equal; or, the Increment of the angle BEC will be equal to the Decrement of the angle BCE; that is, if we ſuppoſe the point *b* indefinitely near to B, the $\angle BEb = \angle BCb$: therefore, if with EB and CB, as radii, the arches B*m* and B*n* be deſcribed; then, becauſe the circumferences of circles are as their radii, $Bm : Bn :: EB : CB$. Deſcribe the circle BGA, whoſe diameter is AB; and in *fig.* 34. produce EB to G, and BC to I. Draw the right lines GA, AI, and IG. Then, in *fig.* 32 and 33. (by 21 E. 3.) $\angle GAI = \angle GBI$ or EBC; and in *fig.* 34. (by 22 E. 3.) $\angle GAI + \angle GBI = 180°. = \angle GBI + \angle EBI$; therefore in this *fig.* alſo, $\angle GAI = \angle EBI$ or EBC. Now, if we ſuppoſe the arches B*m* and B*n* to be little right lines perpendicular to E*m* and C*n* reſpectively, then may the triangles *b*B*m* and BAG be conſidered as ſimilar, as may likewiſe the triangles *b*B*n* and BAI; (for, by 31 E. 3. the angles BGA and BIA are right; and, becauſe the points *b* and B are preſumed to be indefinitely near to each other, therefore the $\angle$ EbA is indefinitely near to equality with the $\angle$ EBA, and the $\angle$CbA with the $\angle$CBA; &c.) wherefore, by 4 E. 6. bB : BA :: Bm : AG, and bB : BA :: Bn:AI; $\because$ Bm : Bn :: AG : AI. Hence EB : CB :: AG : AI; and therefore, by 6 E. 6. the triangles EBC and GAI are alike, and the $\angle$ AGI $=$ $\angle$BEC: but, by 21 E. 3. the $\angle$AGI $=$ $\angle$ABI or ABC; therefore the $\angle$BEC $=$ $\angle$ABC, and conſequently the triangles CAB and BAE are ſimilar; wherefore, by 4 E. 6. CA : AB :: BA : AE; $\because$ AB $=\overline{AC\times AE}\big|^{\frac{1}{2}}$.

Conſtruction.

Produce EA to K, making AK $=$ AC; deſcribe the ſemicircle KE; and, in *fig.* 33 and 34. draw AQ perpendicular to the diameter EK, and make AB equal to AQ: then will B be the point required. For, by 35 E. 3. AQ $=\overline{KA\times AE}\big|^{\frac{1}{2}}$, that is, AB $=$ $\overline{AC\times AE}\big|^{\frac{1}{2}}$.

EXAMPLE X.

56. Given the point C in the radius EA of the circle AB &c. To find the point B, Fig. 35.

at which, if a tangent TB and right line CB be drawn to it, the Angle CBT is a *Minimum*.

Draw the radius EB. Now, becauſe by 18 E. 3. the tangent to a circle is always perpendicular to the radius, it is evident, that, the angle CBT will be the leaſt poſſible when the angle CBE is the greateſt: It is likewiſe evident, that, when the ſaid angle CBE is the greateſt poſſible, the Fluxions of the angles BCE and BEC will be equal; or, ſuppoſing the point b indefinitely near to B, $\angle BCb = \angle BEb$; and therefore, if with CB, as a radius, we deſcribe the little circular arch Bn; then, CB : Bn :: EB : Bb, or, CB : BE :: nB : Bb; and conſequently, by 6 E. 6. (becauſe $\angle CBn = \angle EBb$, and therefore $\angle CBE = \angle nBb$,) the triangles CBE and nBb are ſimilar, and $\angle ECB = \angle bnB =$ a right angle. Hence, when the angle CBE is a Maximum, or the angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA.

Or, Becauſe by Trigonometry EB : ſine $\angle$ BCE :: EC : ſine $\angle$ CBE; therefore, the angle CBE will be the greateſt when the angle ECB is a right one; but, when the angle CBE is the greateſt, the angle CBT is the leaſt poſſible: therefore, when the ſaid angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA; as before.

EXAMPLE XI *.

57. Given the point C in the radius OA of the ſemicircle ABE: To find the point B in the ſaid ſemicircle where the Sum of the right lines CB and EB is a *Maximum*. *Fig.* 36.

Suppoſe the point *b* indefinitely near to B; and the little circular arches B*n* and B*m*, deſcribed with CB and EB as radii, to be little right lines perpendicular to the ſaid radii reſpectively. Now, the angles CB*n*, OB*b*, and EB*m*, being right; therefore $\angle CBO = \angle nBb$, and $\angle OBE = \angle mBb$: but, when $CB + EB$ is a Maximum, it is evident, that, the Increment *nb* muſt be = the Decrement *bm*; and therefore the triangles *bn*B and *bm*B are equal and ſimilar. Hence, $\angle nBb$ being $= \angle mBb$, therefore $\angle CBO = \angle OBE = \angle OEB$; and the triangles OCB and BCE are ſimilar; and conſequently, by 4 E. 6. OC : CB :: BC : CE; $\therefore CB = \overline{CO \times CE}|^{\frac{1}{2}}$.

Con-

---

* Invented *Anno* 1761.

Conſtruction.

Make CK=CO; deſcribe the ſemicircle KQE, and draw the right line CQ perpendicular to the diameter EK; alſo, make CB =CQ: then will B be the point required. For, by 35 E. 3. $CQ=\overline{EC\times CK}|^{\frac{1}{2}}$, that is, $CB=\overline{EC\times CO}|^{\frac{1}{2}}$.

Corollary:

In order to make the Maximum take place, the given diſtance OC muſt be greater than $\frac{1}{3}$ of the radius OA.

Note.

To find an Algebraical expreſſion for the ſum of the right lines CB and EB when the ſaid ſum is a Maximum. Put the radius EO or $OA=a$, and $OC=b$: then, $CB=\overline{CO\times CE}|^{\frac{1}{2}}=\overline{ab+b^2}|^{\frac{1}{2}}$, and CB : BO :: CE : EB, or $EB=\frac{BO\times CE}{CB}=\frac{a^2+ab}{\overline{ab+b^2}|^{\frac{1}{2}}}$: therefore

$$CB+EB=\overline{ab+b^2}|^{\frac{1}{2}}+\frac{a^2+ab}{\overline{ab+b^2}|^{\frac{1}{2}}}=\frac{\overline{a+b}|^2}{b^{\frac{1}{2}}.\overline{a+b}|^{\frac{1}{2}}}$$

$$=\overline{\frac{\overline{a+b}|^3}{b}}\Big|^{\frac{1}{2}}.$$

Ex-

EXAMPLE XII*.

58. Let the given right line CG in it's firſt ſituation coincide with the right line AE; and let the end G move along the right line ED in ſuch a manner that the other end C may paſs with an uniform motion from A to E; and, at the ſame time, ſuppoſe a point B to move with the ſame uniform motion from C along the line CG; then, by the motion of this point, will the Curve ABD be deſcribed: To find the point C in the line AE where CF is a *Maximum*; BF being always parallel to DE, and BH parallel to AE. Fig. 37.

Put $AE=ED=CG=a$, $AF=x$, and $AC=CB=z$; then, $CF=x-z$, $BG=a-z$, and $FE=BH=a-x$. Now, by 4 E. 6. $GB : BH :: BC : CF$, that is, $a-z : a-x :: z : x-z$; $\therefore ax-az-xz+z^2=az-xz$, or $ax-2az+z^2=0$; the Fluxion of which equation is $a\dot{x}-2a\dot{z}+2z\dot{z}=0$: but, when CF or $x-z$ is a Maximum, it is evident, that, $\dot{x}=\dot{z}$; therefore, by dividing by $\dot{x}$ or $\dot{z}$, we have $a-2a+2z=0$, or $2z=a$; and therefore $z=\frac{1}{2}a$, or, the point C biſects the line AE when CF is a Maximum.

Corollary.

---

* Invented *Anno* 1756.

Corollary.

Since $CB = BG$; therefore $CF = BH = FE = \frac{1}{2}CB$; and conſequently, when the angle AED is not an Obtuſe but a Right one, the angle BCF will, when CF is a Maximum, be $= 60^{\circ}$.

EXAMPLE XIII.

Fig. 38. 59. **To find the point of Retrogreſſion B in the *Contracted Semicycloid* ABD; whoſe generating ſemicircle AGF is greater than it's baſe FD*.**

Put the generating ſemicircle $AGF = a$, baſe $FD = b$, radius $OG = c$, $CG = s$, $OC = x$,

or

---

* This Curve may be thus generated.—Let the ſemicircle *af* roll along upon the right line *fd* equal to it and perpendicular to it's diameter *fa*: then will the curve ABD deſcribed by any point A taken *without* the ſaid ſemicircle and in the radius O *a* produced, be a *Contracted Semicycloid*. For, deſcribe the concentric ſemicircle AGF; in any poſition of which, as BRK, to the generating point B draw the ordinate CB parallel to the baſe FD, which ſaid baſe muſt evidently be equal and parallel to *fd*: through the center *o* draw MR parallel to the diameter AF; and with the radius *o*B deſcribe the arch BM: then will the arches AG, MB, and RK, be equal, and $tB = CG$; and therefore $GB = Ct = fr =$ (by the generation) arch *rk*; but, ſemicircle AF : ſemicircle *af* :: arch RK : arch *rk*; therefore, ſemicircle AF : *fd* or baſe FD :: arch AG : GB; which is the property of the *Cycloid*. (See *art.* 35.)

This Curve may, with propriety, be called an *exterior* Cycloid.

ordinate CB=y, and arch AG=z; and let the point g be ſuppoſed indefinitely near to G, and ng parallel to CF; that is, let nG =s', and Gg=z'. Now, if the Increment Gg be ſuppoſed a little right line perpendicular to the radius OG, the right angled triangles OCG and Gng will be ſimilar; and therefore, by 4 E. 6. CO : OG :: nG : Gg, that is, $x : c :: s' : z'$, or (*art.* 7.) $x : c :: \dot{s} : \dot{z} = \frac{c\dot{s}}{x}$. By the known property of the curve, $a : b :: z :$ GB$=\frac{bz}{a}$; therefore CG+GB$=s+\frac{bz}{a}=y$, which, at the point of Retrogreſſion, muſt evidently be a Maximum; it's Fluxion therefore is =0, that is, becauſe the Fluxions of s and z are negative to each other, $-\dot{s}+\frac{b\dot{z}}{a}=0$; from which equation we have $\dot{z}=\frac{a\dot{s}}{b}$. Hence $\frac{c\dot{s}}{x}=\frac{a\dot{s}}{b}$; and therefore $ax=bc$, and $x=\frac{bc}{a}=$OC.

Or, * Put the generating ſemicircle AGF =a, baſe FD=b, radius OG=c, and OC =x; and let gb be ſuppoſed indefinitely near neat and parallel to GB. Now, it is evident, that, at the point of Retrogreſſion B, the tangent muſt be perpendicular to the ordinate; that is, at the ſaid point, the Increment Bb is perpendicular to bv; and therefore, becauſe the $\angle tBb = \angle oBv$, the right angled triangles Bbv and Bto or GCO are ſimilar; and conſequently, by 4 E. 6. Bv : vb :: GO : OC. But, by the nature or generation of the curve, a : b :: Gg or Bv : vb. Hence therefore, a : b :: GO : OC, that is, $a : b :: c : x$; $\because x = \frac{bc}{a} = OC$; as before.

---

* Invented *Anno* 1760.

Corollaries.

1. At any point B in the curve, if to the correſponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making TG : GB :: a : b, that is, if TG be made equal to the arch AG; or if Bx be made equal to the radius Oa, and xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B, be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be ſimilar; and therefore, &c.

2. If the right line *a*Q be drawn perpendicular to the radius OA; then, when the point Q arrives at the baſe FD, the point A will be in the point of Retrogreſſion B. For, ſince at the ſaid point of Retrogreſſion, $x=\frac{bc}{a}$; by analogy, $a:b::c:x$; that is, ſemicircle BRK : ſemicircle *prk* :: *o*R : *ot*; therefore, the ſemicircles being as their radii *o*R and *or*, the point *t* coincides with *r*, that is, the ordinate CB coincides with the right line *fd*; and the triangles R*op* and B*ot* are equal and ſimilar. Conſequently, R*p* is perpendicular to *po*; and therefore, &c.—Hence the following

Conſtruction.

Make DR=arch *ae*; draw R*t* equal and parallel to F*f*, and *t*B equal and parallel to *a*Q: then will B be the point of Retrogreſſion required.

CHAP. V.

Of finding the Points of Inflection in Curves.

60. WHEN a Curve from being Concave becomes Convex towards it's axis; or, from

being Convex becomes Concave; then, that Point in it where the Change is made, or that which ſeparates the Convex from the Concave part, is called the Point of *Inflection.* So that, if to the Point of Inflection a Tangent be drawn, it will cut the Curve.

Thus, in *fig.* 39. if AB be concave and BY convex, or, in *fig.* 40. if AB be convex and BY concave towards the axis AX; then B is the point of Inflection: where the Tangent TBG cuts the curve. *Fig.* 39. 40.

61. Now, it is evident, that, in any Curve, in order to determine whether the Abſciſs or Ordinate flows with an accelerated or retarded motion, or, to find the value of it's *Second Fluxion*, it is neceſſary that one of them be made to increaſe or decreaſe with a given or uniform motion, with which the ſwiftneſs of the increaſe or decreaſe of the other may be always compared.

Suppoſe the Abſciſs therefore, it being the moſt natural, always to flow equably, or equal parts of it to be deſcribed in equal times; then, becauſe the Direction of the curve from A to B (*fig.* 39.) or from B to Y (*fig.* 40.) continually approaches nearer to a Paralleliſm with the axis, it is evident, the ordinate between theſe points muſt flow with a motion continually Retarded: and, becauſe the Direction of the curve from B to Y (*fig*. 39.) or from A to B (*fig*. 40.) approaches continually nearer to a Perpendicular to the axis; therefore, between theſe points, the ordinate muſt flow, or increaſe, with a motion continually Accelerated. Conſequently, at the point of Inflection, the ordinate will flow with neither an Accelerated nor Retarded but with an Uniform motion: therefore, at the point of Inflection B, the *Second Fluxions* of the abſciſs and ordinate will be $=0$.

Or, Let B*n* be a given right line always parallel to the baſe, B*m* a Tangent to the curve, and *nm* a right line parallel to the ordinate. Then, it is plain, that, before the ordinate arrives at the point of Inflection, the right line *nm*, in *fig*. 39. will be continually Decreaſing, and afterwards continually Increaſing; or, in *fig*. 40. will be continually Increaſing, and afterwards continually Decreaſing: therefore, at the point of Inflection, it will be neither Increaſing nor Decreaſing; but will, in *fig*. 39. be a Minimum, or, in *fig*. 40. a Maximum; and conſequently, in either caſe, it's Fluxion will be $=0$. But, by *art*. 24. the right lines B*n*, *nm*, and B*m*, are as the Fluxions of the abſciſs, ordinate, and curve, reſpectively. Hence therefore, the *Second Fluxions* of the abſciſs and ordinate, at the point of Inflection, are =0; as before.

62. Now, ſince, in general, the ordinate CB and the meaſure of it's Fluxion, *nm*, flow together; or, ſince the Fluxion of the ordinate CB is always as the right line *nm*, and the ſaid right line is a Variable or Flowing quantity *; therefore, to find the *Second Fluxion* of a *Fluent*, or the *Fluxion* of an Expreſſion containing the *Firſt Fluxions* of any variable quantities; every *Firſt* Fluxion, not ſuppoſed Invariable, muſt be conſidered as a diſtinct Variable quantity, and the Expreſ-

---

* If the right line *nm* be invariable; then, the curve will degenerate into a right line, and the ordinate will flow uniformly, or the Fluxion of it be always the ſame.

If the right line *nm* be variable; then, the velocity with which the ordinate flows will likewiſe be variable: And, the velocity with which the line *nm* increaſes or decreaſes will always be as the increaſe or decreaſe of the velocity with which the ordinate flows; that is, the Fluxion of the line *nm* will be as the Second Fluxion of the ordinate CB.

If the right line *nm* does not uniformly increaſe or decreaſe; then, the velocity with which the ordinate flows will not uniformly increaſe or decreaſe: And, the increaſe or decreaſe of the velocity with which the line *nm* increaſes or decreaſes will always be as the increaſe or decreaſe of the acceleration or retardation of the velocity with which the ordinate flows; that is, the Second Fluxion of the line *nm* will be as the Third Fluxion of the ordinate CB.

ſion be put into Fluxions by the *Rules* laid down in *Chap.* 2.

Thus, the Fluxion of $2y\dot{y}$ is $2\dot{y}\dot{y}+2y\ddot{y}$, that is, $2\dot{y}^2+2y\ddot{y}$; or, if $\dot{y}$ be invariable, it is $2\dot{y}^2$. The Fluxion of $\frac{\dot{x}y}{\dot{y}}$ is $\frac{\ddot{x}y\dot{y}+\dot{x}\dot{y}^2-\ddot{y}\dot{x}y}{\dot{y}^2}$; or, ſince either $\dot{x}$ or $\dot{y}$ may be ſuppoſed invariable, (that is, either x or y to flow with an uniform motion,) if we make $\dot{x}$ invariable, it will be $\frac{\dot{x}\dot{y}^2-\ddot{y}\dot{x}y}{\dot{y}^2}$. Alſo, the Fluxion of

$$\frac{x\dot{x}}{\overline{a^2+y^2}|^{\frac{1}{2}}} \text{ is } \overline{\dot{x}^2+x\ddot{x}}\times\overline{a^2+y^2}|^{-\frac{1}{2}}-\frac{y\dot{y}}{\overline{a^2+y^2}|^{\frac{1}{2}}}\times x\dot{x}.$$

THE Third and Fourth Fluxions, &c. are found in the ſame manner, due regard being had to ſuch Fluxions as are ſuppoſed Invariable.

63. HENCE, to find the point of Inflection B; put the Equation of the curve (where the abſciſs AC is $=x$ or $a\pm x$, and the ordinate CB$=y$,) into Fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ or $\dot{y}$; and put this $\dot{x}$ or $\dot{y}$ and it's value into Fluxions, making both $\ddot{x}$ and $\ddot{y}=0$: then, by expunging the reſt of the Fluxional quanti-

ties, you may have x or y, at the point of Inflection ſought, determined.

64. BUT, the point of Inflection may be found without the help of Second Fluxions. For, if TB be a Tangent to it, it is evident, that, AT, the Difference between the ſubtangent and abſciſs, will be a *Maximum*. And therefore, to find the point of Inflection, we need only to find a definitive expreſſion for the Subtangent, by *Chap.* 3. and the Difference between it and the abſciſs; and then make the Fluxion of this Difference =0.—And ſometimes it may be determined in a *New* and different manner; as will be ſhewn in the following Example.

EXAMPLE I.

65. To find the point of *Inflection* B in the *Protracted Semicycloid* ABD; whoſe generating ſemicircle AGF is leſs than it's baſe FD *. *Fig.* 41.

---

* This Curve may be thus generated.—Let the ſemicircle af roll along upon the right line fd equal to it and perpendicular to it's diameter fa; then will the curve ABD deſcribed by any point A taken *within* the ſaid ſemicircle and in the radius Oa, be a *Protracted Semicycloid*. For, deſcribe the concentric ſemicircle AGF; in any poſition of which, as BRK, to the generating point B draw the ordinate CB parallel to the baſe

Put the generating ſemicircle AGF$=a$, baſe FD$=b$, radius OG$=c$, CG$=s$, OC$=x$, ordinate CB$=y$, and arch AG$=z$; and let the point g be ſuppoſed indefinitely near to G, and ng parallel to CF, that is, let Gn $=s'$, $ng=x'$, and gG$=z'$. Now, if the Increment Gg be ſuppoſed a little right line perpendicular to the radius OG, the right angled triangles OCG and Gng will be ſimilar; and therefore, by 4 E. 6. OC : CG :: Gn : ng, that is, $x : s :: s' : x'$, or (*art.* 7.) $x : s :: \dot{s} : \dot{x}$, $\therefore \dot{s}=\frac{x\dot{x}}{s}$. By the known property of the curve, $a : b :: z :$ GB$=\frac{bz}{a}$; therefore CG+GB$=s+\frac{bz}{a}=y$; the Fluxion of which equation, becauſe the Fluxions of s and z are negative to each other, is $-\dot{s}+\frac{b\dot{z}}{a}=\dot{y}$. Hence, $\dot{y}=-\frac{x\dot{x}}{s}+\frac{b\dot{z}}{a}$, that is, (becauſe by

---

baſe FD, which ſaid baſe muſt evidently be equal and parallel to fd: through the center o draw Mr parallel to the diameter af; and with the radius oB deſcribe the arch BM: then will the arches AG, MB, and RK, be equal, and tB$=$CG; and therefore, GB$=$C$t=fr=$ (by the generation) arch rk; but, ſemicircle AF : ſemicircle af :: arch RK : arch rk; therefore, ſemicircle AF : fd or baſe FD :: arch AG : GB; which is the property of the *Cycloid*. (See *art.* 35.)

This Curve may, with propriety, be called an *interior* Cycloid.

by 47 E. 1. $s = \overline{c^2 - x^2}\big|^{\frac{1}{2}}$,) $\dot{y} = -\frac{x\dot{x}}{\overline{c^2 - x^2}\big|^{\frac{1}{2}}} + \frac{b\dot{z}}{a}$. But, by 4 E. 6. $s : c :: x' : z'$; that is, $\overline{c^2 - x^2}\big|^{\frac{1}{2}} : c :: \dot{x} : \dot{z} = \frac{c\dot{x}}{\overline{c^2 - x^2}\big|^{\frac{1}{2}}}$; wherefore, by ſubſtitution, $\dot{y} = \left(-\frac{x\dot{x}}{\overline{c^2 - x^2}\big|^{\frac{1}{2}}} + \frac{bc\dot{x}}{a.\overline{c^2 - x^2}\big|^{\frac{1}{2}}} = \right) \frac{-ax\dot{x} + bc\dot{x}}{a.\overline{c^2 - x^2}\big|^{\frac{1}{2}}}$. Now, the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is $0 =$

$$\frac{-a\dot{x}^2 \times a.\overline{c^2 - x^2}\big|^{\frac{1}{2}} + \frac{ax\dot{x}}{\overline{c^2 - x^2}\big|^{\frac{1}{2}}} \times \overline{-ax\dot{x} + bc\dot{x}}}{a^2.\overline{c^2 - x^2}},$$

that is, (by reduction,) $0 = \frac{-ac^2\dot{x}^2 + bcx\dot{x}^2}{a.\overline{c^2 - x^2}\big|^{\frac{3}{2}}}$; whence, $0 = -ac + bx$, and therefore $x = \frac{ac}{b} = OC$.

Or *, Put the generating ſemicircle AGF $= a$, baſe FD $= b$, radius OG $= c$, and OC $= x$; and let gb be ſuppoſed indefinitely near and parallel to GB. Now, it is evident, that, at the point of Inflection B, the angle made by the tangent and ordinate muſt be a Maximum; that is, at the ſaid point, the

∠Bbv

---

* Invented *Anno* 1760.

$\angle Bbv$ is a Maximum. But, by the nature or generation of the curve, $a : b :: Gg$ or $Bv : vb$; and, by Trigonometry, $bv : vB ::$ s.$\angle vBb$: s.$\angle Bbv$; therefore, the ſaid $\angle Bbv$ is the Greateſt when vB is perpendicular to Bb. Conſequently, at the point of Inflection B, the tangent is coincident with the radius oB; and the triangles Bvb and toB or COG are ſimilar: therefore, by 4 E. 6. $Bv : vb ::$ CO : OG; that is, $a : b :: x : c$; $\because x = \frac{ac}{b} =$ OC; as before.

Corollaries.

1. At any point B in the curve, if to the correſponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making TG : GB :: $a : b$, that is, if TG be made equal to the arch AG; or, if Bx be made equal to the radius Oa, and xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be ſimilar; and therefore, &c.

2. If the right line AQ be drawn perpendicular to the radius Oa; then, when the

point Q arrives at the right line *fd* parallel to the baſe FD, the point A will be in the point of Inflection B. For, ſince at the ſaid point of Inflection, $x = \frac{ac}{b}$; by analogy, $b : a :: c : x$, that is, ſemicircle *prk* : ſemicircle BRK :: *o*B : *ot*; therefore, the ſemicircles being as their radii, *ro* : *o*B :: B*o* : *ot*, that is, *ot* or OC is a third proportional to the radii O*f* and OF, and the triangles *ro*B and B*ot* are ſimilar. Conſequently, *r*B is perpendicular to *po*; and therefore, &c.—Hence the following

Conſtruction.

Make DR = arch *fe*; draw *r*M through the point R equal and parallel to *f*A; make *o*R = OF; and laſtly, deſcribe the ſemicircles MR and *or*: then will the interſecting point B of the ſaid ſemicircles be the point of Inflection required.

EXAMPLE II.

66. To find the point of *Inflection* B in the *Concboid* of *Nicomedes* AB &c.*

Fig. 42. Put PE = a, EA = b, EC = x, and CB = y; then (*art.* 33.) $\dot{y} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \overline{b^2 - x^2}|^{\frac{1}{2}}}$. Now,

the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y}=0$, is (by reduction) $0=\frac{2ab^4x-3ab^2x^3-b^2x^4}{x^4\times\overline{b^2-x^2}|^{\frac{3}{2}}}\dot{x}^2$; therefore, $2ab^2-3ax^2-x^3=0$; by which equation, x, and conſequently the point B, may be determined.—And, if $a=b$, it will be $2a^3-3ax^2-x^3=0$; which divided by $a+x$, makes $2a^2-2ax-x^2=0$; from which quadratic equation we have $x=\overline{3a^2}|^{\frac{1}{2}}-a$.

---

* See the generation of this Curve, *art.* 33. *note.*

Conſtruction.

Make $Pn=\frac{2}{5}PE$; draw the indefinite right line nm perpendicular to Pn; make $Es=\frac{1}{5}EA$; draw the right lines sr and Pm parallel to each other and perpendicular to the right line mr; make $Et=EA$; and parallel to sr draw the right line tC: then an ordinate drawn from the point C will fall on the point of Inflection B. For, by 4 E. 6. $CE : Et :: Pn : nm$, that is, $x : b :: \frac{2}{5}a : \frac{2ab}{5x}=nm$; and $Pn : nm :: mn : nr$, that is, $\frac{2}{5}a : \frac{2ab}{5x} :: \frac{2ab}{5x} : \frac{2ab^2}{5x^2}=nr$; again, $tE : EC :: sE : Er$, that is, $b : x :: \frac{1}{5}b : \frac{1}{5}x=Er$; therefore,, ($nE$ being $=\frac{3}{5}a$) $nr=\frac{3}{5}a+\frac{1}{5}x$. Hence, $\frac{2ab^2}{5x^2}=\frac{3}{5}a+\frac{1}{5}x$, and therefore $2ab^2=3ax^2+x^3$, or $2ab^2-3ax^2-x^3=0$.

Or,

When $a=b$; make Pv=PA, and PC= Ev: then will C be the point in the abſciſs from which the ordinate to the point of Inflection muſt be drawn. For then $\overline{Pv^2-PE^2}|^{\frac{1}{2}}=\overline{3a^2}|^{\frac{1}{2}}=Ev=PC$; and therefore, $EC=\overline{3a^2}|^{\frac{1}{2}}-a=x$.

EXAMPLE III.

67. To find the point of *Inflection* B in the Curve ABY, whoſe Equation (putting the abſciſs AC$=x$, ordinate CB$=y$, and the perpendicular AE$=a$,) is $ax^2=a^2y+x^2y$. Fig. 43.

The Fluxion of the equation of the curve is $2ax\dot{x}=a^2\dot{y}+2x\dot{x}y+x^2\dot{y}$; therefore, $\dot{y}=\frac{2ax\dot{x}-2x\dot{x}y}{a^2+x^2}$, that is, (by writing $\frac{ax^2}{a^2+x^2}$ for y it's value,) $\dot{y}=\frac{2a^3x\dot{x}}{\overline{a^2+x^2}|^2}$; and the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y}=0$, is

$$0=\frac{2a^3\dot{x}^2\times\overline{a^2+x^2}|^2-:\overline{4a^2x\dot{x}+4x^3\dot{x}}\times 2a^3x\dot{x}}{\overline{a^2+x^2}|^4};$$

which multiplied by $\overline{a^2+x^2}|^4$ and divided by $2a^3\dot{x}^2$, gives $0=\overline{a^2+x^2}|^2-4a^2x^2-4x^4$; therefore $4x^4+4a^2x^2=\overline{a^2+x^2}|^2$; which equation divided by x^2+a^2, makes $4x^2=a^2+x^2$; therefore, $3x^2=a^2$, and $x=a\sqrt{\frac{1}{3}}$; and, if this value of x be ſubſtituted for it in the given equation of the curve, we ſhall have y, or, the ordinate at the point of Inflection $=\frac{1}{4}a$.

Conſtruction.

Make $Ae=\frac{1}{3}AE$; and with the radius eE deſcribe the arch EC: then will C be the point from which the ordinate to the point of Inflection muſt be drawn. For then $eC=\frac{2}{3}a$, and by 47E.1. $Ce^2-eA^2=AC^2$, that is, $\frac{4}{9}a^2-\frac{1}{9}a^2=\frac{1}{3}a^2=AC^2=x^2$, or $x=a\sqrt{\frac{1}{3}}$.

Note.

IF it were required to find the Aſymptote to the curve:—Suppoſe the abſciſs and curve to be indefinitely extended: then, becauſe x^2 will be indefinitely near to equality with a^2+x^2, we ſhall have y (which by the equation of the curve is $=a\times\frac{x^2}{a^2+x^2}$,) indefinitely near to equality with the given right line a; that is, y will then be $=a$, *minus* a quantity indefinitely ſmall. Wherefore, if from the point E a right line, EZ, be drawn parallel to the axis, AX, it will be the Aſymptote required.

SCHOLIUM.

68. IN any Curve, in order to know whether it be *concave* or *convex* towards any point aſſigned in the axis; find the value of $\ddot{y}$ at that point: then, (*art.* 61.) if this value of $\ddot{y}$ comes out Poſitive, the curve will be *convex* towards the axis; and if it comes out Negative, it will be *concave.*

Thus, in the laſt Example, if it were required to find whether the Curve be at firſt *concave* or *convex* towards the axis:—Suppoſe $a=10$, and make $x=1$, or $x=$ any number leſs than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$; then, becauſe $\ddot{y}$ may here be conſidered to be always as $\overline{a^2+x^2}\big|^2-4a^2x^2-4x^4$, the expreſſion for $\ddot{y}$ will come out Poſitive; and therefore, the curve is at firſt *convex* towards the axis: And, when $x=6$, or $x=$ any number greater than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$, the ſaid expreſſion will come out Negative; and therefore, then, the curve will be *concave* towards the axis.

CHAP. VI.

Of finding the Radius of Curvature.

69. AS the curvature, or convexity, of all curves, but circles, varies in every point; therefore, if circles are deſcribed coinciding with a given curve in any number of points, the Radii of theſe circles will be different: And the finding of theſe Radii is the buſineſs of this Chapter.

70. AND, becauſe all curves, but circles, are formed or generated, or may be conceived to be formed or generated, by the evolution, or winding off, of ſome other curves; therefore, the centers of the circles which coincide or have equal degrees of curvature with the different Points or rather Increments of the curves thus formed or generated will always be in the curves to be unwound; which curves are called the *Evolutes*; and the others formed or generated, or conceived to be formed or generated, by their evolution, are called the *Involutes*.

71. Thus, let DEF be any curve, called an Evolute; round which conceive a thread to be wound and extended beyond the curve from D in a right line to A; and let this *Fig.* 44.

thread be evolved, or wound off, from the curve DF, ſo that it bé continually ſtretched at it's full length as it leaves the curve: then will the point A generate, or deſcribe, the Involute curve ABY; and the right lines AD, BE, YF, will be the Radii of Curvature at the points A, B, Y, reſpectively.

Corollaries.

1. The radius of curvature BE will always be equal to the length of the curve ED and right line DA: And, conſequently, if the vertical diſtance, or ſhorteſt radius DA, vaniſhes; that is, if the radius at A be nothing; then, the Involute curve will begin at D; and the curve DE will be equal to the radius of curvature at the point B.

2. Becauſe (18 E. 3.) the radius of a circle is perpendicular to the tangent, the radius of curvature at any point B is always perpendicular to a tangent at that point.

3. The radius BE, which is perpendicular to the Involute at the point B, is a tangent to the Evolute at the point E.

PROBLEM.

72. To deduce a General Expreſſion for BE the Radius of Curvature at any point B in the Involute Curve ABY whoſe Axis is AX and Evolute DE.

Put the abſciſs $AC=x$, and ordinate $CB=y$: and ſuppoſe *b*E indefinitely near to BE, *bc* indefinitely near and parallel to BC; and B*m* parallel to AX; that is, let C*c* or $Bn=x'$, and $nb=y'$. Then, B*b* being conſidered as a little right line coinciding with a tangent to the point B, the right angled triangles B*nb* and BCH will be ſimilar; (for $\angle EBb=\angle CBn$, and therefore, the $\angle EBn$ being common, the $\angle$s *n*B*b* and CBH are equal; *ergo*, &c.) and, the $\angle Bbm$ being right, the right angled triangles *mnb* and *bn*B will, by 8 E. 6, be ſimilar alſo. Wherefore, by 4 E. 6, B*n* : *nb* :: BC : CH, that is, $x' : y' :: y : \frac{yy'}{x'}=CH$; and therefore, by 47 E. 1.

$BH=\overline{BC^2+CH^2}|^{\frac{1}{2}}=\overline{y^2+\frac{y^2y'^2}{x'^2}}|^{\frac{1}{2}}=\frac{y}{x'}\times$ $\overline{x'^2+y'^2}|^{\frac{1}{2}}$; alſo, $AH=x+\frac{yy'}{x'}$: Again, B*n* : *nb* :: *bn* : *nm*, that is, $x' : y' :: y' : nm=\frac{y'^2}{x'}$;

$\therefore$ B$m = x' + \frac{y'^2}{x'}$. Now, becauſe the Direction of the curve ABY approaches continually nearer to a Paralleliſm with the axis AX; therefore, if we ſuppoſe the abſciſs (AC$=x$,) to flow with an equable or uniform motion, that is, ſuppoſing x' or $\dot{x}$ to be invariable, or always of the ſame value; then, the Increment of the ordinate (CB$=y$,) or the Velocity or Fluxion with which it flows, muſt continually decreaſe; that is, the *Second Moment* or *Second Fluxion* of y will be negative: and therefore, Hb, the Increment of AH, viz. the Increment of $x + \frac{yy'}{x'}$, will be $= x' + \frac{y'^2 - yy''}{x'}$. Now, the triangles EB$m$ and EHb are evidently ſimilar; therefore, Bm—Hb : Bm :: (BE—HE$=$) BH : BE, that is, $\frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}}$: $\frac{\overline{x'^2 + y'^2}|^{\frac{3}{2}}}{x'y''} =$ BE; or (*art.*7.) BE$= \frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}}$.

Or, With the radius EB deſcribe the circular Arch BK; *Fig.* 45. which Arch will therefore have the ſame Degree of Curvature with the Involute Curve AB at the point B. Draw the radius EK parallel to the axis AX; and produce the ordinate BC to L, to which draw AN parallel. Put the abſciſs AC$=x$, ordinate CB$=y$, radius EB or EK$=r$, KN$=a$, and NA$=b$; then, LE$=r-a-x$. Now, if we ſuppoſe the abſciſs, x, to increaſe uniformly, and B*m* to be a tangent to the point B; then, if we draw *mn* parallel to BC, and B*n* parallel to AX, by *art.* 24. B*n*, *nm*, and *m*B, will be as the *Fluxions* of the abſciſs, ordinate, and curve, reſpectively; that is, B*n* will be as $\dot{x}$, *nm* as $\dot{y}$, and (becauſe by 47 E. 1, $Bm=\overline{Bn^2+nm^2}|^{\frac{1}{2}}$,) B*m* as $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$. Now, the triangles B*nm* and BLE are ſimilar; therefore, by 4 E. 6, B*n* : *nm* :: BL : LE, that is, $\dot{x} : \dot{y} :: y+b : r-a-x$; $\therefore r\dot{x}-a\dot{x}-x\dot{x}=y\dot{y}+b\dot{y}$; the Fluxion of which equation (ſuppoſing $\dot{x}$ invariable, and therefore, the direction of the curve AB continually approaching towards a paralleliſm with it's axis, the Fluxion of $\dot{y}$ as negative,) is $-\dot{x}^2=\dot{y}^2-y\ddot{y}-b\ddot{y}$; $\therefore \dot{x}^2+\dot{y}^2=\overline{b+y}\times\ddot{y}$. Again, by 4 E. 6, LB : BE :: *n*B : B*m*, that is, $b+y : r :: \dot{x} : \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; $\therefore b+y=\frac{r\dot{x}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$; which ſubſtituted for $b+y$ makes the above

$\dot{x}^2+\dot{y}^2=\dfrac{r\dot{x}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{1}{2}}}$; therefore, $\overline{\dot{x}^2+\dot{y}^2}\times$ $\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{1}{2}}=r\dot{x}\ddot{y}$, that is, $\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{3}{2}}=r\dot{x}\ddot{y}$; $\therefore$ $\dfrac{\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{3}{2}}}{\dot{x}\ddot{y}}=r=BE$, as before.

73. *Note.* When the abſciſs, x, flows with an uniform motion; it follows from *art.* 61. that the ordinate, y, flows with a retarded motion when it increaſes and the curve is concave, or when it decreaſes and the curve is convex towards the axis; and with an accelerated motion when it decreaſes and the curve is concave, or when it increaſes and the curve is convex towards the axis. Now, when y increaſes with an accelerated motion, or it's *Second Fluxion* is affirmative, the General Expreſſion for the Radius of Curvature will be $=\dfrac{\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$; where the Negative ſign ſhews it's poſition.

74. HENCE, becauſe we may ſubſtitute 1 for any invariable Fluxion*; if we put

---

* The ſubſtituting *unity*, or 1, for an invariable Fluxion, has no other effect than it's making the operation leſs laborious; and, in reality, it is no more than making *unity* the Standard of the other Fluxions, or reducing the other Fluxions to a compariſon with 1.

$\dot{x}=1$, the General Expreſſion for the Radius of Curvature will be $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when y increaſes with a Retarded motion, or it's *Second Fluxion* is Negative; and $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ when y increaſes with an Accelerated motion, or it's *Second Fluxion* is Affirmative. The former takes place when the curve is Concave, and the latter when it is Convex towards the axis.——Wherefore, if we put the Equation of the given curve, expreſſing the relation between the abſciſs x and ordinate y, into Fluxions, making $\dot{x}=1$; or, from the nature of the curve, find the value of $\dot{x}=1$ in terms of x, y, and $\dot{y}$; and then put this *fluxional* Equation into Fluxions again, ſtill ſubſtituting 1 for $\dot{x}$, and making the Fluxion of $\dot{y}$ Negative when the curve is Concave, and Affirmative when it is Convex towards the axis; from thence the Values of the *ſecond* and *ſquare* of the *firſt Fluxions* of y may be determined: which being ſubſtituted for them in one of theſe two general expreſſions, viz. in the former when the Fluxion of $\dot{y}$ is Negative, and in the latter when it is Affirmative; we ſhall have a definitive expreſſion for BE, that is, an expreſſion for it free from Fluxions, or, the Radius of Curvature required.

Note.

75. THE Vertical Diſtance, or Radius AD, may be obtained, by writing for $\dot{x}$ and $\dot{y}$ their values, in the General Expreſſion for the Subnormal CH, which was found in *art.* 72. $(=\frac{yy'}{x'}$, that is, by writing $\dot{x}$ for x', and $\dot{y}$ for y',) $=\frac{y\dot{y}}{\dot{x}}$; or, making $\dot{x}=1$, by ſubſtituting the value of $\dot{y}$ in $y\dot{y}$, that is, by multiplying the value of $\dot{y}$ by y; and then making y vaniſh in the definitive expreſſion which will then be found.——For, the expreſſion for the Subnormal CH being the ſame at whatever point in the curve B is taken; therefore, if it be taken at A, where y vaniſhes or becomes $=0$, the point C muſt of conſequence coincide with the vertex A, and the points E and H with D: therefore, &c.

EXAMPLE I.

Fig. 46. 76. To find the Radius of Curvature at any point B in the *Parabola* AY.

Put the parameter $=a$, abſciſs AC$=x$, and ordinate CB$=y$. Now, by a well known property of the curve, $ax=y^2$; the Fluxion of which equation is $a\dot{x}=2y\dot{y}$, or, making $\dot{x}=1$, it is $a=2y\dot{y}$; therefore, $\dot{y}=\frac{a}{2y}=\frac{a}{2\times\overline{ax}|^{\frac{1}{2}}}$, for y is $=\overline{ax}|^{\frac{1}{2}}$ by the equation of the curve: And, the Fluxion of this equation again, (the direction of the curve approaching continually towards a paralleliſm with the axis, and therefore the Fluxion of $\dot{y}$ being negative, *art.* 61.) is $-\ddot{y}=\frac{-a^2}{4\times\overline{ax}|^{\frac{3}{2}}}$: Whence, $\dot{y}^2=\frac{a^2}{4\times ax}=\frac{a}{4x}$, and $\ddot{y}=\frac{a^2}{4\times\overline{ax}|^{\frac{3}{2}}}$. Now, if for $\dot{y}^2$ and $\ddot{y}$ we ſubſtitute theſe their values, we ſhall have $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$, the general expreſſion for the Radius of Curvature BE (*art.* 74.) $=\frac{\overline{1+\frac{a}{4x}}|^{\frac{3}{2}}\times 4.\overline{ax}|^{\frac{3}{2}}}{a^2}$ $=\frac{\overline{4ax+a^2}|^{\frac{3}{2}}}{2a^2}$.

Conſtruction.

Through the point B deſcribe the ſemicircle ABn; biſect Cn in H; make Hr=

$2AC$; and drop the perpendicular rE, terminated by the right line BE drawn through the point H; then will BE be the Radius of Curvature at the point B. For, by 35 E. 3. $BC^2=AC\times Cn$, or $\frac{BC^2}{AC}=Cn$, that is, $\frac{ax}{x}=a=Cn$, and therefore $CH=\frac{1}{2}a$, and $Cr=\frac{1}{2}a+2x$: by 47 E. 1. $\overline{CH^2+CB^2}|^{\frac{1}{2}}=BH$, that is, $\overline{\frac{1}{4}a^2+ax}|^{\frac{1}{2}}=BH$; and by 4 E. 6. $CH : HB :: Cr : BE$, that is, $\frac{1}{2}a : \overline{\frac{1}{4}a^2+ax}|^{\frac{1}{2}} :: \frac{1}{2}a+2x : \frac{a+4x}{a}\times\overline{\frac{1}{4}a^2+ax}|^{\frac{1}{2}}=\frac{\overline{a^2+4ax}|^{\frac{3}{2}}}{2a^2}=BE$.

Note.

77. By writing for $\dot{y}$ it's value $\frac{a}{2y}$, in $y\dot{y}$, (*art.* 75.) we have $\frac{a}{2}$ or $\frac{1}{2}a=AD$ the vertical Diſtance. Which ſame truth may be inferred from the expreſſion for the Radius BE; for, when the ſaid Radius becomes the vertical Diſtance, that is, when the point B coincides with A, x vaniſhes; and therefore, by ſtriking $4ax$ out of the ſaid expreſſion, we have $\frac{\overline{a^2}|^{\frac{3}{2}}}{2a^2}=\frac{1}{2}a$; as before.

Ex-

EXAMPLE II.

78. To find the Radius of Curvature at any point B in the *Cycloid* ABD *. *Fig.* 47.

Put the radius OF or OD $=a$, abſciſs AC $=x$, ordinate CB $=y$, ſine IG $=s$, and arch FG $=z$. Now, by 35 E. 3. IG $=\overline{DI\times IF}|^{\frac{1}{2}}$, that is, $s=\overline{2ay-y^2}|^{\frac{1}{2}}$; the Fluxion of which equation is $\dot{s}=\frac{a\dot{y}-y\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$: And by the nature of the Cycloid, (*art.* 35.) arch DG $=$ GB, and therefore, arch FG $=$ GI $+$ AC, or AC $=$ arch FG $-$ GI, that is, $x=z-s=$ (by ſubſtituting for s it's above value,) $z-\overline{2ay-y^2}|^{\frac{1}{2}}$; and the Fluxion of this equation, making $\dot{x}=1$, is $1=\dot{z}+\frac{y\dot{y}-a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$.

But (*art.* 72.) $\dot{z}=\overline{\dot{s}^2+\dot{y}^2}|^{\frac{1}{2}}=$ (by writing for $\dot{s}$ it's above value,) $\overline{\frac{\overline{a\dot{y}-y\dot{y}}|^2}{2ay-y^2}+\dot{y}^2}|^{\frac{1}{2}}=\frac{a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$; which ſubſtituted for $\dot{z}$ makes the above equation $1=\frac{a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}+\frac{y\dot{y}-a\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$; that is,

$1=$

---

* See how this Curve may be generated, *art.* 35. *note.*

$1 = \frac{y\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}$; therefore $\dot{y} = \frac{\overline{2ay-y^2}|^{\frac{1}{2}}}{y}$; and the Fluxion of this equation (the Fluxion of $\dot{y}$ being negative,) is $-\ddot{y} =$

$$\frac{\frac{ay\dot{y}-y^2\dot{y}}{\overline{2ay-y^2}|^{\frac{1}{2}}}-\dot{y}.\overline{2ay-y^2}|^{\frac{1}{2}}}{y^2} = \frac{-a\dot{y}}{y.\overline{2ay-y^2}|^{\frac{1}{2}}}$$

= (by writing for $\dot{y}$ it's equal,) $-\frac{a}{y^2}$; that is, $\ddot{y} = \frac{a}{y^2}$. Now, by ſubſtituting $\frac{2ay-y^2}{y^2}$ for $\dot{y}^2$, and $\frac{a}{y^2}$ for $\ddot{y}$, we have, by *art.* 74. $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}} =$

$$\frac{\overline{1+\frac{2ay-y^2}{y^2}}|^{\frac{3}{2}} \times y^2}{a} = \frac{\overline{2ay}|^{\frac{3}{2}}}{ay} = 2.\overline{2ay}|^{\frac{1}{2}} = BE$$

the Radius of Curvature required.

Conſtruction.

Make FH=GB; and through the point H draw the right line BE, making BH=HE=chord GF: then will BE be the Radius of Curvature at the point B. For, *art.* 35. a tangent to the point B is parallel to the chord DG, and by *art.* 71. *corol.* 2. the radius of curvature is always perpendicular to the tangent; therefore, becauſe by 31 E. 3. the ∠DGF is right, BE muſt be parallel to the chord GF. Now, by 4 and 8 E. 6. DF : FG :: GF : FI or CB, ∴ GF $= \overline{DF \times CB}|^{\frac{1}{2}} = \overline{2ay}|^{\frac{1}{2}}$, and $2GF = 2.\overline{2ay}|^{\frac{1}{2}} = BE$.

Note.

79. By *art.* 75. if we multiply the value of $\dot{y}$, viz. $\frac{\overline{2ay - y^2}|^{\frac{1}{2}}}{y}$, by y; we ſhall have the Subnormal $CH = \overline{2ay - y^2}|^{\frac{1}{2}}$; which, when y vaniſhes, becomes $= 0$, and equal to the vertical Diſtance: ſo that the Vertices of the Evolute and Involute curves coincide.

EXAMPLE III.

80. To find the Radius of Curvature at any point B in the Curve AD; whoſe nature is ſuch, that, the Triangle CBT, made of the Ordinate, Tangent, and Subtangent, is always proportional to the Ordinate CB; or, whoſe Subtangent CT is equal to a given line $= a$. (See *art.* 36.) *Fig.* 48.

Put GC $= x$, and CB $= y$; then, by *art.* 25. $\frac{\dot{x}y}{\dot{y}} = a$, that is, if $\dot{x}$ be made $= 1$, $\frac{y}{\dot{y}} = a$;

∵ $\dot{y} = \frac{y}{a}$; therefore, $\dot{y}^2 = \frac{y^2}{a^2}$, and (becauſe here y flows with an accelerated motion, or, it's Second Fluxion is affirmative,) $\ddot{y} = \frac{\dot{y}}{a}$, that is, (by ſubſtituting $\frac{y}{a}$ for $\dot{y}$ it's value,) $\ddot{y} = \frac{y}{a^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ theſe their values, in $\frac{\overline{1+\dot{y}^2}\big|^{\frac{3}{2}}}{-\ddot{y}}$, (*art.* 74.) we have $\frac{\overline{1+\frac{y^2}{a^2}}\big|^{\frac{3}{2}} \times a^2}{-y} = \frac{\overline{a^2+y^2}\big|^{\frac{3}{2}}}{-ay} =$ the Radius of Curvature ſought: Where the Negative ſign, only ſhews, that, the Evolute and Radius of Curvature lie on the other ſide of the curve with regard to x and y.

Conſtruction.

Draw B*n* parallel to CT, T*n* perpendicular to TB, *n*E perpendicular to *n*B, and BE parallel to T*n*: then will BE be the Radius of Curvature at the point B. For, by 47 E. 1. $BT = \overline{TC^2+CB^2}\big|^{\frac{1}{2}} = \overline{a^2+y^2}\big|^{\frac{1}{2}}$, and by 4 E. 6. CT : TB :: TB : B*n*, that is, $a : \overline{a^2+y^2}\big|^{\frac{1}{2}} :: \overline{a^2+y^2}\big|^{\frac{1}{2}} : \frac{a^2+y^2}{a} = Bn$; and, CB : BT :: *n*B :

; BE, that is, $y : \overline{a^2+y^2}\big|^{\frac{1}{2}} :: \frac{a^2+y^2}{a} : \frac{\overline{a^2+y^2}\big|^{\frac{3}{2}}}{ay}$ $=$ BE.

EXAMPLE IV.

81. To find the Radius of Curvature at any point B in the Curve AD; whoſe nature is ſuch, that the Tangent BT is every-where equal to a given line $=a$. *Fig.* 49.

Put GC $=x$, and CB $=y$; then, by 47 E. 1. $\overline{TB^2-BC^2}\big|^{\frac{1}{2}} = CT$, that is, $\overline{a^2-y^2}\big|^{\frac{1}{2}} = CT$ $=$ (*art.* 25.) $\frac{\dot{x}y}{\dot{y}}$; or, making $\dot{x}=1$, $\overline{a^2-y^2}\big|^{\frac{1}{2}} = \frac{y}{\dot{y}}$; $\because \dot{y} = \frac{y}{\overline{a^2-y^2}\big|^{\frac{1}{2}}}$; therefore $\dot{y}^2 = \frac{y^2}{a^2-y^2}$; and (becauſe the Fluxion of $\dot{y}$ is affirmative,)

$$\ddot{y} = \frac{\dot{y} \times \overline{a^2-y^2}\big|^{\frac{1}{2}} + \frac{y^2\dot{y}}{\overline{a^2-y^2}\big|^{\frac{1}{2}}}}{a^2-y^2},$$

that is, by ſubſtituting for $\dot{y}$ it's value, $\ddot{y} = \frac{y+\frac{y^3}{a^2-y^2}}{a^2-y^2} = \frac{a^2y}{\overline{a^2-y^2}\big|^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ theſe their values, in $\frac{\overline{1+\dot{y}^2}\big|^{\frac{3}{2}}}{-\ddot{y}}$, (*art.* 74.) we have $\frac{\overline{1+\frac{y^2}{a^2-y^2}}\Big|^{\frac{3}{2}}\times\overline{a^2-y^2}\big|^{2}}{-a^2y}=\frac{\overline{a^2}\big|^{\frac{3}{2}}\times\overline{a^2-y^2}\big|^{\frac{1}{2}}}{-a^2y}$ $=-\frac{a}{y}\times\overline{a^2-y^2}\big|^{\frac{1}{2}}=$ the Radius of Curvature required: Where the Negative ſign ſhews it's poſition.

Conſtruction.

On the extremity of the ſubtangent, T, erect the perpendicular TE; and draw the right line BE perpendicular to the tangent TB: then will BE be the Radius of Curvature at the point B; or, the point E will be in the Evolute curve. For, the triangles CBT and BTE will be ſimilar; and therefore, by 4 E. 6. BC : CT :: TB : BE, that is, $y:\overline{a^2-y^2}\big|^{\frac{1}{2}}::a:\frac{a}{y}\times\overline{a^2-y^2}\big|^{\frac{1}{2}}=$BE.

82. THE General Expreſſion for the Radius of Curvature found in *art.* 74. being only for Curves referred to an Axis; we ſhall now deduce one for *Spirals*, or thoſe Curves whoſe ordinates are referred to a fixed or central Point.

83. LET CBY be the Curve; C the central point, or that from which all the ordinates iſſue; and BE the Radius of Curvature *Fig.* 50. at the point B, that is, let the point E be ſuppoſed in the Evolute curve: conceive C*b* and E*b* indefinitely near to CB and EB, that is, let the points B and *b* be ſuppoſed indefinitely near to each other; and let CF and C*f* be perpendicular to EB and E*b* reſpectively: then will the points F and *r* be indefinitely near to a coincidence; and therefore, *art.* 7. B*r* and C*r* may be taken as equal to BF and CF. Now, if with the ordinate CB, as a radius, the little circular arch B*n* be deſcribed and conſidered as a little right line perpendicular to C*b*; and the Increment B*b* be conſidered as coinciding with a tangent to the point B; then, the little right angled triangle B*nb* will be ſimilar to the right angled triangle BFC; (for $\angle CBn = \angle EBb$; and therefore, $\angle EBn$ being common, the $\angle CBF = \angle nBb$; and conſequently, the angles at F and *n* being right, $\angle BCF = \angle Bbn$;) therefore, by 4 E. 6. $bB : Bn :: CB : BF$; that is, (if we put the ordinate $CB = y$, $Bn = x'$, and $nb = y'$, when by 47 E. 1. B*b*

will be $=\overline{x'^2+y'^2}|^{\frac{1}{2}}$,) $\overline{x'^2+y'^2}|^{\frac{1}{2}} : x' :: y : \frac{x'y}{\overline{x'^2+y'^2}|^{\frac{1}{2}}} =$ BF or B*r*; And B*b* : *bn* :: BC : CF, that is, $\overline{x'^2+y'^2}|^{\frac{1}{2}} : y' :: y : \frac{yy'}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}$ $=$ CF or C*r*; the Increment of which is *rf*; that is, (ſuppoſing x' to be invariable) $\frac{\overline{y'^2+yy''}\times\overline{x'^2+y'^2}|^{\frac{1}{2}} - \frac{y'y''\times yy'}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}}{x'^2+y'^2}$ $=\frac{x'^2y'^2+y'^4+yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}}=rf$. Again, the triangles EB*b* and E*rf* being ſimilar, B*b*—*rf* : B*b* :: (BE—*r*E, or) *r*B : BE; that is, $\overline{(x'^2+y'^2}|^{\frac{1}{2}} - \frac{x'^2y'^2+y'^4+yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}} =) \frac{x'^4+x'^2y'^2-yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}}$ $: \overline{x'^2+y'^2}|^{\frac{1}{2}} :: \frac{x'y}{\overline{x'^2+y'^2}|^{\frac{1}{2}}} : \frac{y\times\overline{x'^2+y'^2}|^{\frac{3}{2}}}{x'^3+x'y'^2-yx'y''}$ $=$BE; or, *art.* 7. BE$=\frac{y\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}$; which is a General Expreſſion for the Radius of Curvature of all Curves referred to a fixed or central Point, when x' or $\dot{x}$ is invariable.

84.

84. HENCE, if $\dot{x}$ be made $=1$, the General Expreſſion for the Radius of Curvature will be $= \dfrac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$. ——. Wherefore, if we put the Equation of the given Spiral into Fluxions, (making $\dot{x}=1$,) and put this *fluxional* Equation into Fluxions again; and from thence, or from the nature of the curve, find the values of $\dot{y}^2$ and $\ddot{y}$: then, if for $\dot{y}^2$ and $\ddot{y}$ we ſubſtitute theſe their values, in this General Expreſſion, we ſhall have BE the Radius of Curvature required: As in the following Examples.

EXAMPLE I.

85. To find the Radius of Curvature at any point B in the *Spiral* of *Archimedes*, CB &c*. *Fig.* 51.

Put the circumference of the generating circle AF &c. $=a$, and it's radius CA $=b$; ordinate CB $=y$, arch AF $=z$. Let Cf be ſuppoſed indefinitely near to CF, that is, let the ∠FCf be ſuppoſed indefinitely ſmall; and with the ordinate CB, as a radius, deſcribe the little circular arch Bn, which put $=x'$;

---

* See how this Curve is generated, *art.* 39. *note.*

$=x'$; alſo, put $Ff=z'$. Now, by the nature of the curve, $a:b::z:y$, or $z=\frac{ay}{b}$, the Fluxion of which equation is $\dot z=\frac{a\dot y}{b}$: And, by the ſimilar ſectors CB*n* and CF*f*, $y:x'::b:z'=\frac{bx'}{y}$, or, *art.* 7. $\dot z=\frac{b\dot x}{y}$. Hence $\frac{a\dot y}{b}=\frac{b\dot x}{y}$; that is, (making $\dot x=1$,) $\frac{a\dot y}{b}=\frac{b}{y}$; from which equation we have $\dot y=\frac{b^2}{ay}$; therefore $\dot y^2=\frac{b^4}{a^2y^2}$, and $\ddot y=\frac{-ab^2\dot y}{a^2y^2}=$ (by writing for $\dot y$ it's value,) $\frac{-b^4}{a^2y^3}$: And, if we ſubſtitute for $\dot y^2$ and $\ddot y$ theſe their values, we ſhall have $\frac{y\times\overline{1+\dot y^2}|^{\frac{3}{2}}}{1+\dot y^2-y\ddot y}$ (*art.* 84.)$=$

$$\frac{y\times\overline{1+\frac{b^4}{a^2y^2}}\Big|^{\frac{3}{2}}}{1+\frac{b^4}{a^2y^2}+\frac{b^4}{a^2y^2}}=\frac{\overline{a^2y^2+b^4}|^{\frac{3}{2}}}{a^3y^2+2ab^4}=$$ BE, the Radius of Curvature ſought.

Cor-

Conſtruction.

Through the center C draw the indefinite right line Hv perpendicular to the ordinate CB; draw the tangent TB, perpendicular to which draw BH; produce BC to V, making BR=TH, and RV=CH; with BV and BR, as radii, deſcribe the arches Vv and Rr; draw the right line vB; and from the interſecting point r draw rE parallel to vH: then will BE be the Radius of Curvature at the point B. For, (*art.* 39.) CT$=\frac{yz}{b}$, that is, by ſubſtituting $\frac{ay}{b}$ for z, CT$=\frac{ay^2}{b^2}$; and by 8, and 4 E. 6. TC : CB :: CB : CH, that is, $\frac{ay^2}{b^2} : y :: y : \frac{b^2}{a}=$CH; therefore TH=BR $=$B$r=\frac{ay^2}{b^2}+\frac{b^2}{a}$, and BV=B$v=\frac{ay^2}{b^2}+\frac{2b^2}{a}$; and, by 47 E. 1. HB$=\overline{BC^2+CH^2}|^{\frac{1}{2}}=$ $\overline{y^2+\frac{b^4}{a^2}}|^{\frac{1}{2}}=\frac{\overline{a^2y^2+b^4}|^{\frac{1}{2}}}{a}$: Again, by 4 E. 6. B$v$: BH :: Br : BE, that is, $\frac{ay^2}{b^2}+\frac{2b^2}{a}$:

$$\frac{\overline{a^2y^2+b^4}|^{\frac{3}{2}}}{a} :: \frac{ay^2}{b^2}+\frac{b^2}{a} : \frac{\overline{a^2y^2+b^4}|^{\frac{1}{2}}}{a} \times \frac{a^2y^2+b^4}{a^2y^2+2b^4}$$

$$= \frac{\overline{a^2y^2+b^4}|^{\frac{3}{2}}}{a^3y^2+2ab^4} = \mathrm{BE}.$$

EXAMPLE II.

Fig. 52. 86. To find the Radius of Curvature at any point B in the *Logarithmic Spiral* CBY; whose Equation (putting the ordinate CB $=y$, curve CB $=z$, and a and b for two given quantities,) is $az=by$. (See *art.* 40.)

The Fluxion of the equation of the curve is $a\dot{z}=b\dot{y}$; therefore $\dot{z}=\frac{b\dot{y}}{a}$. Let the angle BC$b$ be supposed indefinitely small; and with the ordinate CB, as a radius, let the little circular arch Bn be described. Now, if we consider Bn as a little right line perpendicular to Cb, and Bb as a little right line coinciding with a tangent to the point B; then, by 47 E. 1. $\mathrm{B}b=\overline{\mathrm{B}n^2+nb^2}|^{\frac{1}{2}}$, that is, (putting $\mathrm{B}n=x'$, $nb=y'$, and $\mathrm{B}b=z'$,) $z'=\overline{x'^2+y'^2}|^{\frac{1}{2}}$, or, by substituting the Fluxion for the Increment, $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$, that is, (if we put $\dot{x}=1$,) $\dot{z}=\overline{1+\dot{y}^2}|^{\frac{1}{2}}$. Hence

$\frac{b\dot{y}}{a} = \overline{1+\dot{y}^2}\big|^{\frac{1}{2}}$; which equation ſquared is $\frac{b^2\dot{y}^2}{a^2} = 1+\dot{y}^2$; and this produces $\dot{y}^2 = \frac{a^2}{b^2-a^2}$; therefore, $\dot{y} = \frac{a}{\overline{b^2-a^2}\big|^{\frac{1}{2}}}$; and, this being an invariable quantity, therefore $\ddot{y}=0$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ theſe their values, the general expreſſion for the Radius of Curvature, viz. $\frac{y\times\overline{1+\dot{y}^2}\big|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$ *art.* 84. will become =

$$\frac{y\times\overline{1+\frac{a^2}{b^2-a^2}}\Big|^{\frac{3}{2}}}{1+\frac{a^2}{b^2-a^2}-0} = y\times\overline{\frac{b^2}{b^2-a^2}}\Big|^{\frac{1}{2}} = y\times\frac{b}{\overline{b^2-a^2}\big|^{\frac{1}{2}}}$$

= BE, the Radius of Curvature required.

Conſtruction.

Draw the tangent TB; perpendicular to which draw the right line BE, terminated by the ſubtangent TC produced: then will the point E be in the Evolute curve; or, the right line BE will be the Radius of Curvature at the point B. For then (TE being perpendicular to the ordinate CB,) by 8 E. 6. the triangles CTB and CBE will be ſimilar;

and

and therefore, by 4 E. 6. CT : TB :: CB : BE; that is, (becauſe by *art.* 40. CT : TB :: $\overline{b^2-a^2}|^{\frac{1}{2}} : b$,) $\overline{b^2-a^2}|^{\frac{1}{2}} : b :: y : y \times \frac{b}{\overline{b^2-a^2}|^{\frac{1}{2}}}$ = BE.

CHAP. VII.

Of finding the Nature of the Evolute of a given Involute Curve.

AS it is abſolutely neceſſary for the Learner to be well acquainted with the foregoing Chapter before he enters upon this; we ſhall not here define the meaning of Evolute and Involute curves, it being ſufficiently explained therein.

Fig. 53. 87. LET BE be the Radius of Evolution (or Curvature) at any point B in the Involute curve AB, whoſe abſciſs is AC = x, and ordinate CB = y. Parallel to HA draw EN; produce BC to L; and, equal and parallel to CL, draw DN from the vertex of the Evolute DE. Then will the triangles BHC and BEL be ſimilar; and therefore, by 4 E. 6. BH : HC :: BE : EL, that is, (by *art.* 72 and 7.) $\frac{y}{\dot{x}} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} : \frac{y\dot{y}}{\dot{x}} :: \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}} : \dot{y} \times \frac{\dot{x}^2+\dot{y}^2}{\dot{x}\ddot{y}}$ = EL; and HC : CB :: EL : LB, that is, $\frac{y\dot{y}}{\dot{x}} : y :: \dot{y} \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y}} : \frac{\dot{x}^2 + \dot{y}^2}{\ddot{y}} = \text{LB}$. Now, theſe are General Expreſſions for EL and LB, when $\dot{x}$ is conſidered as invariable, and the Fluxion of $\dot{y}$ as negative. Hence therefore,

88. IF $\dot{x} = 1$, and the Fluxion of $\dot{y}$ be negative; the General Expreſſion for BL will be $= \frac{1 + \dot{y}^2}{\ddot{y}}$, and this multiplied by $\dot{y}$ is $\dot{y} \times \frac{1 + \dot{y}^2}{\ddot{y}} =$ the General Expreſſion for LE. Now, by help of the Equation of the given Involute curve, exterminate $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, out of theſe expreſſions, as in the preceding Chapter; and, by *art.* 75. find the vertical diſtance AD. Then, if we put the abſciſs of the Evolute DN $= u$, and it's ordinate NE $= v$; by help of theſe two equations, $u = \text{BL} - \text{BC}$, and $v = \text{AC} - \text{AD} + \text{LE}$, we may get the Nature of the Evolute curve DE required.

89. *Note.* If the given Involute be convex towards it's axis, and x and y increaſe together, or the Fluxions of x and $\dot{y}$ be both affirmative; then, the General Expreſſions for BL and LE will be $\frac{1 + \dot{y}^2}{-\ddot{y}}$ and $\dot{y} \times \frac{1 + \dot{y}^2}{-\ddot{y}}$.

reſpectively; wherein, the Negative ſign ſhews, that, the points L and E muſt be taken on the Concave ſide of the Involute curve, that is, on the other ſide of it with regard to x and y.

EXAMPLE I.

90. To find the Nature of the Curve DE, by whoſe Evolution the *Parabola* AB is deſcribed.

Put AC $=x$, CB $=y$; and DN $=u$, NE $=u$. Now, (by *art.* 76.) $\dot{y}=\frac{a}{2\times\overline{ax}|^{\frac{1}{2}}}$, $\dot{y}^2=\frac{a}{4x}$, and $\ddot{y}=\frac{a^2}{4\times\overline{ax}|^{\frac{3}{2}}}$; which values of $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, being ſubſtituted for them, make the General Expreſſion for BL, *viz.* $\frac{1\pm\dot{y}^2}{\ddot{y}}$ (*art.* 88.) become $=\frac{\overline{1+\frac{a}{4x}}\times 4.\overline{ax}|^{\frac{3}{2}}}{a^2}=\frac{\overline{4x\mp a\times ax}|^{\frac{1}{2}}}{a}$; and that for LE, *viz.* $\dot{y}\times\frac{1+\dot{y}^2}{\ddot{y}}=\dot{y}\times\text{BL}=\frac{a}{2\times\overline{ax}|^{\frac{1}{2}}}\times\frac{\overline{4x\mp a\times ax}|^{\frac{1}{2}}}{a}=2x+\frac{1}{2}a$. Hence, u

(=

(=BL—BC) $= \frac{\overline{\overline{4x+a}\times ax}|^{\frac{1}{2}}}{a} -$ (y or) $\overline{ax}|^{\frac{1}{2}}$ $= \frac{\overline{4x.ax}|^{\frac{1}{2}}}{a}$; and (becauſe by *art.* 77. the vertical diſtance AD is $=\frac{1}{2}a$,) v (=AC—AD+LE) $=x-\frac{1}{2}a+2x+\frac{1}{2}a=3x$. Now, the former of theſe two equations produces $x^3=\frac{au^2}{16}$; and, the cube of the latter, divided by 27, is $x^3=\frac{v^3}{27}$: therefore, $\frac{au^2}{16}=\frac{v^3}{27}$, and $\frac{27a}{16}u^2=v^3$; which is the equation of the Curve DE, expreſſing the relation between the abſciſs and ordinate; and, the Equation of the Semicubical Parabola (whoſe parameter is $\frac{27a}{16}$,) being the ſame; therefore, the Evolute DE is a *Semicubical Parabola*, whoſe vertex is D.

EXAMPLE II.

Fig. 54. 91. To find the Nature of the Curve AEP; by whoſe Evolution the *Cycloid* ABD is deſcribed *.

Put AC $=x$, CB $=y$, arch FG $=z$, and OD or OF $=a$; then (*art.*78.) $\dot{y} = \frac{\overline{2ay-y^2}\big|^{\frac{1}{2}}}{y}$, $\dot{y}^2 = \frac{2ay-y^2}{y^2}$, and $\ddot{y} = \frac{a}{y^2}$: Wherefore, (*art.* 88.) BL $= \frac{1+\dot{y}^2}{\ddot{y}} = \frac{\overline{1+\frac{2ay-y^2}{y^2}}\times y^2}{a} = 2y$, and LE $= \dot{y} \times$ BL $= \frac{\overline{2ay-y^2}\big|^{\frac{1}{2}}}{y} \times 2y = 2.\overline{2ay-y^2}\big|^{\frac{1}{2}}$. Hence, if we put the abſciſs AN $=u$, and ordinate NE $=v$, we have u (= BL − CB =) $2y-y=y$, and $v=$ (AC + LE =) $x+2.\overline{2ay-y^2}\big|^{\frac{1}{2}}$, that is, (becauſe, *art.*78. $x=z-\overline{2ay-y^2}\big|^{\frac{1}{2}}$,) $v=z+\overline{2ay-y^2}\big|^{\frac{1}{2}}$; or, (writing u for y it's equal,) $v=z+\overline{2au-u^2}\big|^{\frac{1}{2}}$. Wherefore, the Evolute curve AEP is a Cycloid, and equal to the given Cy-

---

* See in what manner a *Cycloid* is generated, *art.* 35. *note.*

Cycloid ABD. For, let AS=SV=a; then (AN being =FI,) AT=FG=z, and NT =$\overline{2au-u^2}|^{\frac{1}{2}}$=IG; and therefore AT+TN =$z+\overline{2au-u^2}|^{\frac{1}{2}}$, that is, AT+TN=NE; which is the property of the Cycloid: therefore, the Evolute AEP is a Cycloid; and, becauſe AV=FD, therefore the Cycloids AEP and ABD are equal.

Example III.

92. To find the Nature of the Evolute of the Curve AD, whoſe Tangent BT is every-where equal to a given line =a. *Fig.* 55.

Let BE be the Radius of Curvature at the point B; then (*art.* 81.) if a perpendicular be erected on the point T, it will paſs through the point E; wherefore, when the point T coincides with F, that is, when the tangent and ordinate become equal, or the points B and D coincide, the point E will likewiſe coincide with D: conſequently, the Vertex of the Evolute coincides with that of the Involute.

Put GF=b, GC=x, CB=y, DN=u, and NE=v; then, (*art.* 81.) $\dot{x}=\frac{y}{\overline{a^2-y^2}|^{\frac{1}{2}}}$,

$\dot{y}^2 = \frac{y^2}{a^2-y^2}$, and $\ddot{y} = \frac{a^2y}{\overline{a^2-y^2}^{\,2}}$: wherefore, (*art.* 89.) BL $= \frac{1+\dot{y}^2}{-\ddot{y}} = \overline{1+\frac{y^2}{a^2-y^2}} \times - \frac{\overline{a^2-y^2}^{\,2}}{a^2y} = \frac{a^2-y^2}{-y}$, and LE $= \dot{y} \times$ BL $= \frac{y}{\overline{a^2-y^2}^{\,\frac{1}{2}}} \times \frac{a^2-y^2}{-y} = -\overline{a^2-y^2}^{\,\frac{1}{2}}$; that is, (becauſe the Negative ſign only ſhews that the points L and E muſt be taken on the Concave ſide of the Involute curve DA,) BL$=\frac{a^2-y^2}{y}$, and LE$=\overline{a^2-y^2}^{\,\frac{1}{2}}$. Hence we have $u =$ (LB+BC—DF$=$) $\frac{a^2-y^2}{y} + y - a$, which equation gives $y = \frac{a^2}{u+a}$, and therefore $\dot{y} = - \frac{a^2\dot{u}}{\overline{u+a}^{\,2}}$; alſo, $v =$ (GF—GC+CT$=$) $b - x + \overline{a^2-y^2}^{\,\frac{1}{2}}$, and therefore $\dot{v} = -\dot{x} - \frac{y\dot{y}}{\overline{a^2-y^2}^{\,\frac{1}{2}}} =$ (becauſe by *art.* 81. CT$=\frac{\dot{x}y}{\dot{y}} = \overline{a^2-y^2}^{\,\frac{1}{2}}$, or, $\dot{x} = \frac{\dot{y}}{y} \times \overline{a^2-y^2}^{\,\frac{1}{2}}$,)

$-\frac{\dot{y}}{y}\times\overline{a^2-y^2}|^{\frac{1}{2}}-\frac{y\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}}=\frac{-a^2\dot{y}}{y\times\overline{a^2-y^2}|^{\frac{1}{2}}}$, that is, (by writing for y and $\dot{y}$ their above values affected with u and $\dot{u}$,) $\dot{v}=\frac{a\dot{u}}{\overline{u^2+2au}|^{\frac{1}{2}}}$: which is an Equation for the Evolute DE, and is also an Equation of the Catenary curve: therefore, the Evolute DE is the *Catenary* *.

Or, The above Equation of the Evolute may be found thus.—Let E$n=u'$, and $ne=v'$; then, the triangles enE and TBE being similar, we have, by 4 E. 6. en : nE :: TB : BE, that is, $v' : u' :: a : \frac{au'}{v'}=$ BE, or (*art.* 7.) $\frac{a\dot{u}}{\dot{v}}=$BE; but, by 47 E. 1. BE$=\overline{\text{ET}^2-\text{TB}^2}|^{\frac{1}{2}}$, that is, (becauſe ET$=NF=u+a$,) BE$=$ $\overline{\overline{u+a}|^2-a^2}|^{\frac{1}{2}}=\overline{u^2+2au}|^{\frac{1}{2}}$: therefore, $\frac{a\dot{u}}{\dot{v}}=\overline{u^2+2au}|^{\frac{1}{2}}$, and $\dot{v}=\frac{a\dot{u}}{\overline{u^2+2au}|^{\frac{1}{2}}}$; as before.

SCHO-

---

* The *Catenary* is a Curve, as ADB or *adb*, formed by a flexible line or chain hanging freely from two points of ſuſpenſion, A and B, or *a* and *b*, whether the ſaid points be horizontal or not. *Fig.* 56.

SCHOLIUM.

93. The Evolute of a *Spiral*, or indeed of any other Curve, may be deſcribed, by finding the Radii of Curvature at ſeveral points in the Involute: For then we ſhall have as many points in the Evolute; through which, if a Curve line be drawn, it will be the Evolute ſought.

PART II.

CHAP. I.

Of Infinite Series *.

AS the Learner may, perhaps, be unacquainted with *Infinite Series*, the knowledge of which is ſometimes abſolutely neceſſary in order to find the *Fluents* (or flowing quantities) of *Fluxions* expreſſed in a Fractional manner, and of ſuch wherein there are Surds or Radical quantities; and, becauſe in ſome of the following pages the *Fluents* of ſuch *Fluxional* expreſſions are to be found; the adding of this Chapter may therefore not be improper, though it is, in ſome meaſure, foreign to the buſineſs in hand.

PROB.

---

* The Methods of reducing compound expreſſions into *Infinite Series*, by diviſion and extracting of roots, as taught in this Chapter, were invented by the great Inventor of *Fluxions* about the year 1664; who at the ſame time, or rather a little before, invented the celebrated *Binomial Theorem*.

PROB. I.

~~To reduce a compound Fractional expreſſion~~ into an *Infinite Series*; that is, into a number of terms, which, if infinitely continued, ſhall be equal to the given fractional expreſſion.

EXAMPLE I.

94. To reduce $\frac{b}{a+x}$ into an *Infinite Series.*

Place the denominator $a+x$ as a diviſor, and the numerator b as a dividend; and divide, as in common algebraic diviſion, until you have 4, 5, 6, or more terms in the Quotient; after which you may find as many terms as you pleaſe, by only conſidering the *law* of the progreſſion of the terms already found. Thus, the four firſt terms being $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, (ſee the operation below,) the *law* of the continuation of the diviſion, or of the *Series,* is plain; for each Succeeding term is evidently produced by multiplying the Preceding by $-\frac{x}{a}$; and conſequently, the fifth term will be $+\frac{bx^4}{a^5}$, the ſixth term $-\frac{bx^5}{a^6}$, &c.

Operation:

$$
\begin{array}{l}
a+x)\ b\ \ldots\ (\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\text{\&c.}\\
\qquad b+\frac{bx}{a}\\
\hline
\qquad 0-\frac{bx}{a}\\
\qquad -\frac{bx}{a}-\frac{bx^2}{a^2}\\
\hline
\qquad\quad 0+\frac{bx^2}{a^2}\\
\qquad\qquad \frac{bx^2}{a^2}+\frac{bx^3}{a^3}\\
\hline
\qquad\qquad 0-\frac{bx^3}{a^3}\\
\qquad\qquad\quad -\frac{bx^3}{a^3}-\frac{bx^4}{a^4}\\
\hline
\qquad\qquad\qquad 0+\frac{bx^4}{a^4}\\
\qquad\qquad\qquad \text{\&c.}
\end{array}
$$

Or, If we put x before a, in the denominator of the above fractional expreſſion; that is, if the diviſor be placed thus, $x+a$, inſtead of $a+x$; then the Quotient, or *Series*, will be $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}+$ &c.

Whence, the *law* of the continuation of the *Series* may be obſerved as before.

SCHOLIUM.

95. IN General, in order to have a *true* or *converging* Series, or that in which the terms continually Decreaſe, the Greateſt term muſt be placed firſt. Thus, in the above Example, if a be greater than x; then a muſt be the firſt term in the diviſor, and $\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+$ &c. will be the *true* Series: But, if x be greater than a; then x muſt be the firſt term in the diviſor, and $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}+$ &c. will be the *true* Series; the other, then, being a *diverging* one, the terms in it continually Increaſing; and conſequently the farther you go in the *Series* the farther it will be from the truth.

THOUGH it is impoſſible to take any number of terms in the *Series* that ſhall *truly* expreſs the value of the quantity given; yet, in general, a few of the leading terms will be near enough the truth for any purpoſe.

EXAMPLE II.

96. To reduce $\frac{a^2}{a^2+2ax+x^2}$ into an *Infinite Series.*

Operation.

$$a^2+2ax+x^2)\,a^2\ldots\ldots\left(1-\frac{2x}{a}+\frac{3x^2}{a^2}-\frac{4x^3}{a^3}+\text{\&c.}\right.$$

$$a^2+2ax+x^2$$

$$0-2ax-x^2$$

$$-2ax-4x^2-\frac{2x^3}{a}$$

$$0+3x^2+\frac{2x^3}{a}$$

$$3x^2+\frac{6x^3}{a}+\frac{3x^4}{a^2}$$

$$0-\frac{4x^3}{a}-\frac{3x^4}{a^2}$$

$$-\frac{4x^3}{a}-\frac{8x^4}{a^2}-\frac{4x^5}{a^3}$$

$$0+\frac{5x^4}{a^2}+\frac{4x^5}{a^3}$$

Now, from theſe four terms of the ſeries it is eaſy to ſee the *law* of the continuation is ſuch, that, the numerators are the powers

of

of x, whoſe Indices are 1 leſs than the numbers of the terms to which they reſpectively belong, multiplied by the ſaid numbers; that, the denominators are the powers of a, whoſe Indices are the ſame with thoſe of the numerators; and that, the ſigns of the terms are alternately changed. So that, the 5th term is $+\frac{5x^4}{a^4}$, the 6th term is $-\frac{6x^5}{a^5}$; and ſo on.

PROB. II.

To reduce a compound Surd quantity into an *Infinite Series*; that is, to free a compound expreſſion from Surds by throwing it into a number of decreaſing terms, which, if infinitely continued, ſhall be equal to the quantity given.

EXAMPLE I.

97. To reduce $\overline{a^2+4y^2}\big|^{\frac{1}{2}}$ into an *Infinite Series.*

Take the ſquare root of a^2, which is a, for the Firſt term of the *Series*; (ſee the Operation below;) then, this ſquared and ſubtracted from a^2+4y^2, leaves $4y^2$; and this remainder divided by the double of the firſt term, (as in the common arithmetical extraction of the ſquare root,) *viz.* by $2a$, gives $+\frac{2y^2}{a}$ for the Second term of the *Series*; which, with the double of the firſt term, being multiplied by $\frac{2y^2}{a}$ the ſaid ſecond term, gives $4y^2+\frac{4y^4}{a^2}$, and this ſubtracted from $4y^2$ leaves $-\frac{4y^4}{a^2}$, which divided by the double of the two firſt terms of the Series, *viz.* by $2a+\frac{4y^2}{a}$, gives $-\frac{2y^4}{a^3}$ for the Third term of the *Series*; which, with the double of the two firſt terms, *viz.* $2a+\frac{4y^2}{a}$, being multiplied by $-\frac{2y^4}{a^3}$ the ſaid third term, gives $-\frac{4y^4}{a^2}-\frac{8y^6}{a^4}+\frac{4y^8}{a^6}$, and this ſubtracted from $-\frac{4y^4}{a^2}$ leaves $\frac{8y^6}{a^4}-\frac{4y^8}{a^6}$, which divided by the double of the three firſt terms of the Series already found, *viz.* by $2a+\frac{4y^2}{a}-\frac{4y^4}{a^3}$, gives $+\frac{4y^6}{a^5}$ for the Fourth term of the *Series*.——After the ſame manner may be found any number of terms in the Series: And, when the *law* of the progreſſion, or of the continuation of the Series, is diſcovered, the terms may be continued on at pleaſure.

Operation.

$$
\begin{array}{rl}
 & a^2+4y^2\left(a+\frac{2y^2}{a}-\frac{2y^4}{a^3}+\frac{4y^6}{a^5}-\&c.\right. \\
 & \underline{a^2} \\
2a) & 0+4y^2 \\
 & \underline{4y^2+\frac{4y^4}{a^2}} \\
2a+\frac{4y^2}{a}) & 0-\frac{4y^4}{a^2} \\
 & \underline{-\frac{4y^4}{a^2}-\frac{8y^6}{a^4}+\frac{4y^8}{a^6}} \\
2a+\frac{4y^2}{a}-\frac{4y^4}{a^3}) & 0+\frac{8y^6}{a^4}-\frac{4y^8}{a^6} \\
 & \underline{\frac{8y^6}{a^4}+\frac{16y^8}{a^6}-\frac{16y^{10}}{a^8}+\frac{16y^{12}}{a^{10}}} \\
 & 0-\frac{20y^8}{a^6}+\frac{16y^{10}}{a^8}-\frac{16y^{12}}{a^{10}}
\end{array}
$$

Ex-

EXAMPLE II.

98. To reduce $\overline{1-x-x^2}|^{\frac{1}{2}}$ into an *Infinite Series.*

Operation:

$$\begin{array}{rl}
 & 1-x-x^2\left(1-\frac{x}{2}-\frac{5x^2}{8}-\frac{5x^3}{16}-\&c.\right. \\
 & 1 \\
\big) & \overline{0-x-x^2} \\
 & -x+\frac{x^2}{4} \\
2-x\big) & \overline{0\ -\frac{5x^2}{4}} \\
 & -\frac{5x^2}{4}+\frac{5x^3}{8}+\frac{25x^4}{64} \\
2-x-\frac{5x^2}{4}\Big) & \overline{0\ -\frac{5x^3}{8}-\frac{25x^4}{64}} \\
 & -\frac{5x^3}{8}+\frac{5x^4}{16}+\frac{25x^5}{64}+\frac{25x^6}{256} \\
 & \overline{0\ -\frac{45x^4}{64}-\frac{25x^5}{64}-\frac{25x^6}{256}} \\
 & \&c.
\end{array}$$

So that $\overline{1-x-x^2}|^{\frac{1}{2}}$ is $= 1-\frac{x}{2}-\frac{5x^2}{8}-\frac{5x^3}{16}-\&c.$

AND, after the ſame manner may any ſuch common Surd quantity be reduced into an *Infinite Series*: But, with much greater eaſe and expedition,

99. ALL ſorts of *fractional* and *ſurd* Quantities may be reduced into *Infinite Series* by the celebrated

BINOMIAL THEOREM[*]

which is this, *viz.* $\overline{P+PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \frac{m-4n}{5n}EQ + \&c.$ Wherein, $P+PQ$ repreſents the Quantity whoſe Power is to be thrown into an *Infinite Series*; P the Firſt term of that quantity, which, in general, muſt be the Greateſt; Q the Other, or Reſt of the terms, divided by the Firſt; $\frac{m}{n}$ the Index of the power, whether it be affirmative or negative: and $A=$ the Firſt term of the *Series*, $B=$ the Second, $C=$ the Third, $D=$ the Fourth, $E=$ the Fifth, &c. that is,

$A=$

---

[*] The Truth of this Theorem has been demonſtrated by various Writers; the Proof of it is therefore here omitted.

$A = P^{\frac{m}{n}}$, $B = \frac{m}{n} AQ$, $C = \frac{m-n}{2n} BQ$, $D = \frac{m-2n}{3n} CQ$, $E = \frac{m-3n}{4n} DQ$, &c.

The following Examples will explain, and ſhew the great Uſe of this curious and noble Theorem.

EXAMPLE I.

100. To reduce $\overline{a^2+4y^2}\big|^{\frac{1}{2}}$ into an *Infinite Series*.

Here, $P = a^2$, $Q = \frac{4y^2}{a^2}$, $m = 1$, $n = 2$, $A = a$, $B = \frac{2y^2}{a}$, $C = -\frac{2y^4}{a^3}$, $D = \frac{4y^6}{a^5}$, $E = -\frac{10y^8}{a^7}$, &c. Therefore, $\overline{a^2+4y^2}\big|^{\frac{1}{2}} = a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \frac{10y^8}{a^7}$, *&c.*

EXAMPLE II.

101. To reduce $\frac{1}{\overline{a^2-y^2}\big|^{\frac{1}{2}}}$, that is, $\overline{a^2-y^2}\big|^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P = a^2$, $Q = -\frac{y^2}{a^2}$, $m = -1$, $n = 2$, $A = a^{-1} = \frac{1}{a}$, $B = \frac{y^2}{2a^3}$, $C = \frac{3y^4}{8a^5}$, $D = \frac{5y^6}{16a^7}$, $E =$

$E = \frac{35y^8}{128a^9}$, &c. Therefore, $\overline{a^2-y^2}\big|^{-\frac{1}{2}} = \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9}$, *&c.*

EXAMPLE III.

102. To reduce $\frac{1}{1-x}$, that is, $\overline{1-x}\big|^{-1}$, into an *Infinite Series.*

Here, $P=1$, $Q=-x$, $m=-1$; $n=1$; $A=1$, $B=x$, $C=x^2$, $D=x^3$, $E=x^4$, &c. Therefore, $\overline{1-x}\big|^{-1} = 1+x+x^2+x^3+x^4$, *&c.*

EXAMPLE IV.

103. To reduce $\frac{1}{1+x}$, that is, $\overline{1+x}\big|^{-1}$, into an *Infinite Series.*

Here, $P=1$, $Q=x$, $m=-1$, $n=1$, $A=1$, $B=-x$, $C=x^2$, $D=-x^3$, $E=x^4$, &c. Therefore, $\overline{1+x}\big|^{-1} = 1-x+x^2-x^3+x^4$, *&c.*

104. BUT, we may often find the *Series* anſwering to a propoſed Quantity by the following

THEOREM,

viz. $\overline{P+PQ}\big|^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n}Q + \frac{m}{n} \times \frac{m-n}{2n} \times Q^2 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}Q^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}$

$\frac{m-2n}{3n} \times \frac{m-3n}{4n} Q^4 +$ &c. (which, indeed, is the ſame as the former, though differently expreſſed,) with ſtill greater eaſe and expedition; for, in this, no previous deduction is required; and both the numerators and denominators of the fractions $\frac{m}{n}$, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, &c. are Series of numbers in arithmetical progreſſion, which have the ſame common difference n. This will appear by the following Examples.——But, *note*, the Former Theorem is, in general, beſt adapted to ſhew the *Law* of the Series.

EXAMPLE I.

105. To reduce $\frac{1}{\overline{a^2+x^2}|^{\frac{1}{2}}}$, that is, $\overline{a^2+x^2}|^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P = a^2$, $Q = \frac{x^2}{a^2}$, $m = -1$, $n = 2$.

Therefore, $\overline{a^2+x^2}|^{-\frac{1}{2}} = \frac{1}{a} \times : 1 - \frac{x^2}{2a^2} + \frac{3.x^4}{2.4.a^4} - \frac{3.5.x^6}{2.4.6.a^6} + \frac{3.5.7.x^8}{2.4.6.8.a^8} -$ &c. $= \frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7} + \frac{35x^8}{128a^9} -$ &c.

EXAMPLE II.

106. To reduce $\overline{a+x}|^{\frac{5}{3}}$ into an *Infinite Series*.

Here, $P=a$, $Q=\frac{x}{a}$, $m=5$, $n=3$. Therefore, $\overline{a+x}|^{\frac{5}{3}}=a^{\frac{5}{3}}\times:1+\frac{5.x}{3.a}+\frac{5.2.x^2}{3.6.a^2}+\frac{5.2.-1.x^3}{3.6.9.a^3}+\frac{5.2.-1.-4.x^4}{3.6.9.12.a^4}+$ &c. $=a^{\frac{5}{3}}+\frac{5a^{\frac{2}{3}}x}{3}+\frac{5x^2}{9a^{\frac{1}{3}}}-\frac{5x^3}{81a^{\frac{4}{3}}}+\frac{5x^4}{243a^{\frac{7}{3}}}$, &c.

EXAMPLE III.

107. To reduce $\frac{b}{a+x}$, that is, $b\times\overline{a+x}|^{-1}$, into an *Infinite Series*.

Here, $P=a$, $Q=\frac{x}{a}$, $m=-1$, $n=1$; and therefore, $\frac{m}{n}$, $\frac{m}{n}\times\frac{m-n}{2n}$, $\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}$, $\frac{m}{n}\times\frac{m-n}{2n}\times\frac{m-2n}{3n}\times\frac{m-3n}{4n}$, &c. will be -1 and $+1$ alternately; and conſequently $b\times\overline{a+x}|^{-1}$ is $=b\times\frac{1}{a}\times:1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-$ &c. $=\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}+\frac{bx^4}{a^5}-$ &c.

EXAMPLE IV.

108. To reduce $\overline{1-x}|^{\frac{1}{4}}$ into an *Infinite Series.*

Here, $P=1$, $Q=-x$, $m=1$, $n=4$. Therefore, $\overline{1-x}|^{\frac{1}{4}}=1-\frac{1}{4}x+\frac{1.-3}{4.8}x^2-\frac{1.-3.-7}{4.8.12}x^3+\frac{1.-3.-7.-11}{4.8.12.16}x^4-$ *&c.* $=1-\frac{1}{4}x-\frac{3}{4.8}x^2-\frac{3.7}{4.8.12}x^3-\frac{3.7.11}{4.8.12.16.}x^4$ — *&c.*

SCHOLIUM.

THE Sum of any Infinite geometrical ſeries decreaſing, is equal to the ſquare of the firſt term divided by the difference between the firſt and ſecond.

Thus, $a+x+\frac{x^2}{a}+\frac{x^3}{a^2}+$ *&c.* is $=\frac{a^2}{a-x}$; and $a-x+\frac{x^2}{a}-\frac{x^3}{a^2}+$ *&c.* is $=\frac{a^2}{a+x}$. For, (*art.* 94.) if a^2 be divided by $a-x$, and by $a+x$, the Quotients will be theſe infinite ſeries of Terms decreaſing in geometrical proportion continued.

CHAP. II.

Of finding the Fluent of a given Fluxion.

109. THE Buſineſs of the *Direct* Method of Fluxions being to find the Fluxion of a given Fluent, or the Velocity with which a Variable quantity flows at any point or term aſſigned; ſo the Buſineſs of the *Inverſe* Method of Fluxions is to determine the Variable quantity, or Fluent, from that Velocity or Fluxion being given. And this, in General, may be done by the following *Rules*, theſe being the converſe of thoſe delivered in *Part* 1. *Chap.* 2.*

RULE I.

110. To find the Fluent of a Simple Fluxion, or, of that wherein there is no variable quantity and but One fluxional letter.

SUBSTITUTE the variable or flowing letter for it's Fluxion: and you will have the Fluent required.

Thus, the Fluent of $a\dot{x}$ is $= ax$. (*art.* 14.)

RULE

---

* To treat at large on the different ways for finding the *Fluents* of the unbounded variety of Fluxional Expreſſions, would, by far, exceed the limits of an *introductory* Tract: This affair therefore cannot, with propriety, be handled here in ſo very extenſive and copious a manner.

RULE II.

111. To find the Fluent of a compound fluxional expreſſion conſiſting of the products of two or more flowing quantities drawn into their Fluxions; that is, which conſiſts of the Fluxion of each quantity drawn into the other or product of the reſt of the quantities.

MULTIPLY the flowing quantities together: and the Product is the Fluent required.

Thus, the Fluent of $\dot{x}y+x\dot{y}$ is $= xy$; the Fluent of $\dot{x}yz+x\dot{y}z+xy\dot{z}$ is $= xyz$; and the Fluent of $\dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}$ is $= vxyz$. (*art.* 15.)

RULE III*.

112. To find the Fluent of a Fraction like $\frac{\dot{x}v-x\dot{y}}{y^2}$.

DIVIDE the laſt term in the numerator by the Fluxion of the Negative ſquare root of the denominator; then, divide this quotient by the Affirmative ſquare root of the denominator:

---

[*] This *Rule* muſt be uſed with caution, as it is not applicable to fractional expreſſions in *general*.

minator: and you will have the Fluent required.

Thus, the Fluent of $\frac{\dot{x}y-x\dot{y}}{y^2}$ is $=\frac{x}{y}$. (*art.* 17.)

RULE IV.

113. To find the Fluent of an expreſſion compounded of different fluxionary terms connected together by the Signs $+$ and $-$.

FIND the ſeparate Fluents of the different terms; which connect together by the Signs of their reſpective Fluxions: and you will have the Fluent required.

Thus, the Fluent of $a\dot{x}+\dot{x}y+x\dot{y}-\frac{\dot{x}y-x\dot{y}}{y^2}$ is $=ax+xy-\frac{x}{y}$. (*art.* 19.)

RULE V.

114. To find the Fluent of an expreſſion which conſiſts of the Fluxion of a variable quantity drawn into any Power of that quantity contained any number of times *.

1°. In Simple Expreſſions,——Strike out the *fluxional* letter; add 1 to the Index of the Power; and divide by the Index thus increaſed: and you will have the Fluent required.

---

* The *Rule* fails when the Index of the Power is -1. To find the Fluent then, ſee *Rule* 6.

Thus, the Fluent of $2x^2\dot{x}$ is $= \frac{2}{3}x^2$; the Fluent of $-\frac{\dot{x}}{x^2}$, that is, of $-x^{-2}\dot{x}$, is $= -x^{-2+1}$ divided by $-2+1$, that is, $= \frac{-x^{-1}}{-1} = x^{-1} = \frac{1}{x}$. And, Univerſally, the Fluent of $mx^{m-1}\dot{x}$ is $= x^m$; and the Fluent of $\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$ is $= x^{\frac{m}{n}}$.

2°. In Compound Expreſſions,—where the *fluxionary* part is equal, or in an invariable ratio, to the Fluxion of the quantity under the vinculum,—Add 1 to the Index of the Power; and divide by the Fluxion of the quantity under the vinculum, drawn into the Index of the Power thus increaſed: and you will have the Fluent required.

Thus, the Fluent of $\overline{x+x^2}|^2 \times \overline{3\dot{x}+6x\dot{x}}$ is $= \frac{\overline{x+x^2}|^3 \times \overline{3\dot{x}+6x\dot{x}}}{3\times\overline{\dot{x}+2x\dot{x}}} = \overline{x+x^2}|^3$; and the Fluent of $-\frac{2}{3}\times\overline{x-a}|^{-\frac{5}{3}}\times\dot{x}$ is $= \frac{-\frac{2}{3}\times\overline{x-a}|^{-\frac{5}{3}+1}\times\dot{x}}{\overline{-\frac{5}{3}+1}\times\dot{x}} = \frac{-\frac{2}{3}\times\overline{x-a}|^{-\frac{2}{3}}\dot{x}}{-\frac{2}{3}\dot{x}} = \overline{x-a}|^{-\frac{2}{3}}$. (*art.* 20.)

RULE

RULE VI.

115. To find the Fluent of a compound fluxional expreſſion, like $\frac{b}{a+x}\dot{x}$, or $\frac{b\dot{x}}{\overline{a^2+x^2}\rceil^{\frac{1}{2}}}$; *&c.*

1°. THROW the expreſſion into an Infinite Series; and find the Fluent of the Series by the foregoing Rules: and you will have the Fluent required.

Thus, to find the Fluent of $\frac{b}{a+x}\dot{x}$; throw the expreſſion into a Series; which (*art.* 94.) is $= \frac{b\dot{x}}{a} - \frac{bx\dot{x}}{a^2} + \frac{bx^2\dot{x}}{a^3} - \frac{bx^3\dot{x}}{a^4} +$ *&c.* and then find the Fluent of this Series; which, by the preceding Rules, is $= \frac{bx}{a} - \frac{bx^2}{2a^2} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4}$ $+$ *&c.* and is the Fluent required.

And, to find the Fluent of $\frac{b\dot{x}}{\overline{a^2+x^2}\rceil^{\frac{1}{2}}}$; throw the expreſſion into a Series; which, (*art.* 105.) is $= b\dot{x} \times : \frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7} +$ *&c.* $=$ $\frac{b\dot{x}}{a} - \frac{bx^2\dot{x}}{2a^3} + \frac{3bx^4\dot{x}}{8a^5} - \frac{5bx^6\dot{x}}{16a^7} +$ *&c.* Now, the

Fluent of this Series, by the foregoing Rules, is $= \frac{bx}{a} - \frac{bx^3}{6a^3} + \frac{3bx^5}{40a^5} - \frac{5bx^7}{112a^7} + \&c.$ which is the Fluent required.

2°. OR, Becauſe (*art.* 21.) the Fluxion of the Hyperbolic Logarithm of any quantity is equal to the Fluxion of that quantity, divided by the quantity itſelf; therefore,

The Fluent of $b \times \frac{\dot{x}}{a+x}$ is $= b \times$ Hyp. Log. of $a+x$. For, the Fluxion of $a+x$ is $\dot{x}$, which divided by $a+x$ is $\frac{\dot{x}}{a+x}$.

And the Fluent of $b \times \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ is $= b \times$ Hyp. Log. of $x+\overline{a^2+x^2}|^{\frac{1}{2}}$. For, the Fluxion of $x+\overline{a^2+x^2}|^{\frac{1}{2}}$ is $\dot{x} + \frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} =$

$$\frac{\dot{x} \times \overline{a^2+x^2}|^{\frac{1}{2}} + x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} = \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} \times : x+\overline{a^2+x^2}|^{\frac{1}{2}},$$

which divided by $x+\overline{a^2+x^2}|^{\frac{1}{2}}$ is $\frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$.

&c.

SCHOLIUM.

116. THOUGH the Fluxion of any Fluent, how much ſoever compounded it be, may be accurately found; yet, the Fluents of compound Fluxional expreſſions cannot always be had in finite terms.

117. THOUGH no Fluent can have more than One Fluxion; yet, a Fluxion may have an Infinite number of Fluents. Thus, for Example, the Fluent of $\dot{x}$ may be either x or $x \pm a$; wherein, a repreſents any invariable quantity whatſoever.—Now, to find a, when it muſt be added to or taken from the Fluent x, is called *correcting* the Fluent: And to effect this; in any Equation, after having obtained the Fluent of each ſide by the foregoing Rules; make the variable letter in either of them vaniſh, or equal to nothing; and ſubſtitute for the variable letter in the other, ſuch a determinate or invariable value as it is then known to have: or, for the variable quantities, write ſuch invariable or fixed values as they are reſpectively known to have at any particular point or term. Then, if we ſubtract the ſides of this new Equation from the correſponding Fluents before found; the remaining Fluents will

be Contemporary, or always equal to each other; and conſequently, we ſhall then have the correct Fluent ſought determined.—— This affair may, perhaps, be better underſtood by giving the following Examples.

Example 1.

To find the Correct Fluent of $\dot{y}=2x\dot{x}$.

The Fluent of this equation, by *art.* 114. is $y=x^2$.—Now, when $y=0$, if $x=0$; then, $y-0=x^2-0$, or $y=x^2$: therefore, the Fluent firſt found needs no correction.

Example 2.

To find the Correct Fluent of $a\dot{x}-2x\dot{x}=2y\dot{y}$.

The Fluent of this equation, by *art.* 114. is $ax-x^2=y^2$. Now, when y ends, or when $y=0$, if x be $=a$; then, ſubſtituting 0 for y, and a for x, the fluential equation will become $a^2-a^2=0$, that is $0=0$: therefore, the Fluent firſt found needs no correction.

Example 3.

To find the Correct Fluent of $\dot{z}=a\dot{y}$.

The Fluent of this equation, by *art.* 110. is $z=ay$. Now, if y be $=b$ when z is $=0$;

then, by writing in this fluential equation a for z and b for y, it will become $0=ab$; therefore, ay is always ab greater than z; and conſequently, the Fluent corrected is $z=ay-ab$.

Example 4.

To find the Correct Fluent of $\dot{y}=-\overline{a+x}\vert^{-3}\times 2\dot{x}$.

The Fluent of this equation, by *art.* 114. is $y=\frac{-\overline{a+x}\vert^{-2}\times 2\dot{x}}{-2\dot{x}}=\overline{a+x}\vert^{-2}=\frac{1}{\overline{a+x}\vert^{2}}$. Now, if $y=0$ when $x=0$; then, this fluential equation will become $0=\frac{1}{a^2}$: and therefore, y is always leſs than $\frac{1}{\overline{a+x}\vert^{2}}$ by the quantity $\frac{1}{a^2}$: conſequently, the Contemporary Fluents will be $y=\frac{1}{\overline{a+x}\vert^{2}}-\frac{1}{a^2}$.

Example 5.

To find the Correct Fluent of $\dot{z}=b\times\frac{\dot{x}}{\overline{a^2+x^2}\vert^{\frac{1}{2}}}$.

The Fluxion of the Hyp. Log. of $x+\overline{a^2+x^2}\vert^{\frac{1}{2}}$ is $=\frac{\dot{x}}{\overline{a^2+x^2}\vert^{\frac{1}{2}}}$ (*art.* 21. or 115.)

therefore, the Fluent of $\dot{z}=b\times\frac{\dot{x}}{\overline{a^2+x^2}\big|^{\frac{1}{2}}}$ is $z=b\times$ Hyp. Log. of $x+\overline{a^2+x^2}\big|^{\frac{1}{2}}$. Now, when $z=0$, if x be likewiſe $=0$, this Fluent will then become $0=b\times$Hyp. Log. of a; which ſubtracted from the ſaid Fluent, makes the Correct Fluent, or *true* value of $z=b\times$ Hyp. Log. of $x+\overline{a^2+x^2}\big|^{\frac{1}{2}}-b\times$ Hyp. Log. of a (which, by the Nature of Logarithms, is) $=b\times$ Hyp. Log. of $\frac{x+\overline{a^2+x^2}\big|^{\frac{1}{2}}}{a}$.

Example 6.

To find the Correct Fluent of $a\dot{x}-bx\dot{x}=y\dot{y}$.

The Fluent of this equation (*art.* 114.) is $ax-\frac{1}{2}bx^2=\frac{1}{2}y^2$. Now, if $x=c$ when $y=d$; then, ſubſtituting c for x and d for y, the equation will be $ac-\frac{1}{2}bc^2=\frac{1}{2}d^2$; and therefore, by ſubtracting the correſponding ſides of this equation from the above, we ſhall have the Correct or Contemporary Fluents $ax-\frac{1}{2}bx^2-ac+\frac{1}{2}bc^2=\frac{1}{2}y^2-\frac{1}{2}d^2$.

CHAP.

CHAP. III.

Of finding the Length of a Curve Line.

Fig. 57. 58. 118. IN Curves referred to an Axis, (*fig.* 57.) let cb be ſuppoſed indefinitely near and parallel to the ordinate CB, and Bn equal and parallel to Cc the Increment of the abſciſs AC: And in Curves referred to a fixed or central Point, (*fig.* 58.) let bC be ſuppoſed indefinitely near to BC, and the indefinitely little circular arch Bn be deſcribed with the ordinate or radius CB. Put AC (*fig.* 57.) $=x$, CB$=y$, curve AB$=z$; Bn $=x'$, $nb=y'$, and B$b=z'$: then, (the Increment Bb being conſidered as a little right line,) by 47 E. 1. $Bb=\overline{Bn^2+nb^2}|^{\frac{1}{2}}$, that is, $z'=\overline{x'^2+y'^2}|^{\frac{1}{2}}$, or (*art.* 7.) $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$. Put the ordinate CB (*fig.* 58.) $=y$, curve CPB $=z$, B$n=x'$, $nb=y'$, and B$b=z'$: then, (becauſe Bb may be conſidered as an indefinitely ſmall right line, and Bn as a little right line perpendicular to Cb,) as before, $z'=\overline{x'^2+y'^2}|^{\frac{1}{2}}$, or $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$. And this is a General Expreſſion for the Fluxion of the Length of any Curve Line whatſoever *.

---

* The ſame expreſſion may be derived, without the help of *Increments*, from *art.* 24 and 38.

Now, by help of the Equation or Properties of the given Curve whoſe Length is required, we may find the value of $\dot{x}^2$ in terms of $\dot{y}^2$, or of $\dot{y}^2$ in terms of $\dot{x}^2$; and then, by ſubſtitution, $\dot{x}^2$ or $\dot{y}^2$ in this General Expreſſion will be exterminated: and by finding the Fluent of the reſulting equation, we ſhall have the value of z, or the Length of the Curve required.

EXAMPLE I.

119. To find the Length of the Curve AB $=z$, whoſe Equation (putting the given line AG $=\frac{2}{3}a$, GC $=x$, and CB $=y$,) is $2\times\overline{a^2+x^2}\big|^{\frac{3}{2}}=3a^2y$. *Fig.* 59.

The Fluxion of this equation is $3\times\overline{a^2+x^2}\big|^{\frac{1}{2}}\times 2x\dot{x}=3a^2\dot{y}$; therefore, $\dot{y}=\frac{2x\dot{x}}{3a^2}\times 3\times\overline{a^2+x^2}\big|^{\frac{1}{2}}$ $=\frac{2x\dot{x}}{a^2}\times\overline{a^2+x^2}\big|^{\frac{1}{2}}$, and $\dot{y}^2=\frac{4x^2\dot{x}^2}{a^4}\,\overline{a^2+x^2}$ $=\frac{4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}$; which ſubſtituted for $\dot{y}^2$, makes $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}\big|^{\frac{1}{2}}$ (*art.* 118.) $=$ $\overline{\frac{a^4\dot{x}^2+4a^2x^2\dot{x}^2+4x^4\dot{x}^2}{a^4}}\bigg|^{\frac{1}{2}}=\frac{a^2\dot{x}+2x^2\dot{x}}{a^2}=$ the Fluxion of the Curve AB; whoſe Fluent is

$z=$

$z = \frac{a^2x + \frac{2}{3}x^3}{a^2} = x + \frac{2x^3}{3a^2} =$ the Length of the Curve AB required.

EXAMPLE II.

120. To find the Length of the common *Cycloid**.

Put OA or OF the radius of the generating circle $=a$, abſciſs AC$=x$, ordinate CB $=y$, CG$=s$, and arch AB$=z$; then, (as
Fig. 60. was found in *art.* 35.) $\dot{y} = \frac{2a-x}{s}\dot{x}$, and therefore $\dot{y}^2 = \frac{\overline{2a-x}|^2}{s^2}\dot{x}^2$; which ſubſtituted for $\dot{y}^2$, makes the general expreſſion for the Fluxion of the Curve (*art.* 118.) $\dot{z} = \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} = \overline{\dot{x}^2 + \frac{\overline{2a-x}|^2}{s^2}\dot{x}^2}\Big|^{\frac{1}{2}} = \overline{\frac{s^2\dot{x}^2+4a^2\dot{x}^2-4ax\dot{x}^2+x^2\dot{x}^2}{s^2}}\Big|^{\frac{1}{2}}$, that is, (becauſe by 35 E. 3. GC^2=AC×CF, or $s^2 = 2ax-x^2$,) $\dot{z} = \overline{\frac{4a^2\dot{x}^2-2ax\dot{x}^2}{2ax-x^2}}\Big|^{\frac{1}{2}} = \overline{\frac{2a\dot{x}^2}{x}}\Big|^{\frac{1}{2}} = \overline{2a}|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$; and the Fluent of this is $z=$

---

* See the generation of this Curve, *art.* 35. *note.*

$z = \overline{2a}|^{\frac{1}{2}} \times 2x^{\frac{1}{2}} = 2 \times \overline{2ax}|^{\frac{1}{2}} =$ twice the chord AG; (for, the triangles FAG and GAC being ſimilar, FA : AG :: GA : AC, or $\overline{FA \times AC}|^{\frac{1}{2}} =$ AG, that is, $\overline{2ax}|^{\frac{1}{2}} =$ AG.) Whence, by writing $2a$ for x, the Length of the Semicycloid AD appears to be equal to twice the diameter AF of it's generating Semicircle.

EXAMPLE III.

121. To find the Length of a *Parabola*.

Put the parameter $= a$, abſciſs AC $= x$, ordinate CB $= y$, and curve AB $= z$; then (*art.* 28.) $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; which ſubſtituted for $\dot{x}^2$, makes $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (the general expreſſion for the Fluxion of the Length of the Curve, *art.* 118.) $= \overline{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{\dot{y}}{a} \times \overline{a^2 + 4y^2}|^{\frac{1}{2}}$; which, thrown into an Infinite Series, (*art.* 100.) is $= \frac{\dot{y}}{a} \times : a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} -$ &c. that is, $\dot{z} = \dot{y} + \frac{2y^2\dot{y}}{a^2} - \frac{2y^4\dot{y}}{a^4} + \frac{4y^6\dot{y}}{a^6} -$ &c. And the Fluent of this Series *Fig.* 61.

is $z=y+\frac{2y^3}{3a^2}-\frac{2y^5}{5a^4}+\frac{4y^7}{7a^6}-$ &c. = the Length of the Curve AB required.

Or, The above $\dot{z}=\frac{\dot{y}}{a}\times\overline{a^2+4y^2}\big|^{\frac{1}{2}}$ is $=$

$$\frac{\dot{y}\times\overline{a^2+4y^2}}{a\times\overline{a^2+4y^2}\big|^{\frac{1}{2}}}=\frac{a^2y\dot{y}+4y^3\dot{y}}{a\times\overline{a^2y^2+4y^4}\big|^{\frac{1}{2}}}=\frac{\frac{1}{2}a^2y\dot{y}+4y^3\dot{y}}{a\times\overline{a^2y^2+4y^4}\big|^{\frac{1}{2}}}$$

$$+\frac{\frac{1}{2}a^2\dot{y}}{a\times\overline{a^2+4y^2}\big|^{\frac{1}{2}}}=\frac{1}{a}\times\overline{a^2y^2+4y^4}\big|^{-\frac{1}{2}}\times$$

$$\overline{\tfrac{1}{2}a^2y\dot{y}+4y^3\dot{y}}+\tfrac{1}{4}a\times\frac{\dot{y}}{\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}}.$$

Now, (*art.* 114.) the Fluent of the firſt of theſe two terms is $=\frac{1}{2a}\times\overline{a^2y^2+4y^4}\big|^{\frac{1}{2}}=\frac{y}{a}\times\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}$; and the Fluent of the laſt of the ſaid two terms (*art.* 115.) is $=\frac{1}{4}a\times$ Hyp. Log. of $y+\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}$; therefore, $z=\frac{y}{a}\times\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}$ $+\frac{1}{4}a\times$ Hyp. Log. of $y+\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}$. But, ſince when z and y vaniſh or become $=0$, (as at the vertex A,) this Fluent becomes $=$ $\frac{1}{4}a\times$ Hyp. Log. of $\frac{1}{2}a$; therefore the ſaid Fluent being Corrected (*art.* 117. *Ex.* 5.) makes the *true* value of z or the Length of the Curve AB $=\frac{y}{a}\times\overline{\frac{1}{4}a^2+y^2}\big|^{\frac{1}{2}}+\frac{1}{4}a\times$ Hyp.

Log. of $y+\overline{\tfrac{1}{4}a^{2}+y^{2}}^{\frac{1}{2}}-\tfrac{1}{4}a\times$ Hyp. Log. of $\tfrac{1}{2}a=\frac{y}{a}\times\overline{\tfrac{1}{4}a^{2}+y^{2}}^{\frac{1}{2}}+\tfrac{1}{4}a\times$ Hyp. Log. of $\frac{y+\overline{\tfrac{1}{4}a^{2}+y^{2}}^{\frac{1}{2}}}{\tfrac{1}{2}a}$, by the Nature of Logarithms.

EXAMPLE IV.

122. To find the Length of any Arch of a *Circle*.

Put the radius EA$=a$, abſciſs AC$=x$, ordinate CB$=y$, arch AB$=z$: then, (*art.* 27.) $\dot{x}=\frac{y\dot{y}}{a-x}=$ (becauſe $a-x=$ CE $=\overline{\mathrm{EB}^{2}-\mathrm{BC}^{2}}^{\frac{1}{2}}=\overline{a^{2}-y^{2}}^{\frac{1}{2}}$,) $\frac{y\dot{y}}{\overline{a^{2}-y^{2}}^{\frac{1}{2}}}$; therefore $\dot{x}^{2}=\frac{y^{2}\dot{y}^{2}}{a^{2}-y^{2}}$; which ſubſtituted for $\dot{x}^{2}$, makes $\dot{z}=\overline{\dot{x}^{2}+\dot{y}^{2}}^{\frac{1}{2}}$ (the general expreſſion for the Fluxion of the Length of the Curve, *art.* 118.) $=\overline{\frac{y^{2}\dot{y}^{2}}{a^{2}-y^{2}}+\dot{y}^{2}}^{\frac{1}{2}}=\frac{a\dot{y}}{\overline{a^{2}-y^{2}}^{\frac{1}{2}}}=a\dot{y}\times\overline{a^{2}-y^{2}}^{-\frac{1}{2}}$; which thrown into an Infinite Series (*art.* 101.) is $\dot{z}=a\dot{y}\times:\frac{1}{a}+\frac{y^{2}}{2a^{3}}+$ *Fig.* 62.

$$\frac{3y^4}{8a^5}+\frac{5y^6}{16a^7}+\frac{35y^8}{128a^9}+\frac{63y^{10}}{256a^{11}}+\frac{231y^{12}}{1024a^{13}}+\frac{429y^{14}}{2048a^{15}}+\&c.=\dot{y}+\frac{y^2\dot{y}}{2a^2}+\frac{3y^4\dot{y}}{8a^4}+\frac{5y^6\dot{y}}{16a^6}+\frac{35y^8\dot{y}}{128a^8}+\frac{63y^{10}\dot{y}}{256a^{10}}+\frac{231y^{12}\dot{y}}{1024a^{12}}+\frac{429y^{14}\dot{y}}{2048a^{14}}+\&c.$$

And the Fluent of this Series (*art.* 114.) is

$$z=y+\frac{y^3}{6a^2}+\frac{3y^5}{40a^4}+\frac{5y^7}{112a^6}+\frac{35y^9}{1152a^8}+\frac{63y^{11}}{2816a^{10}}+\frac{231y^{13}}{13312a^{12}}+\frac{143y^{15}}{10240a^{14}}+\&c.=$$

the Length of the Arch AB,

Now, if we ſuppoſe the radius EA $=a$ $=1$, and the ∠AEB=30°. then, (becauſe the ſine of any arch is equal to half the chord of twice that arch, and the chord of 60°. is equal to the radius,) the ſine, or ordinate CB=y, will be $=\frac{1}{2}$; and therefore, the terms of the above Fluent being reduced into Decimal Fractions, and placed under one another, will ſtand thus; viz.

.500000000

.500000000 &c.
.020833333
.002343750
.000348772
.000059339
.000010923
.000002118
.000000426
&c.

The ſum of which is .5235987 &c. = the Length of the Arch AB:
which × by 12

is 6.283185 &c. = the Periphery of a Circle whoſe Radius is 1;
therefore 3.141592 &c. = the Periphery of a Circle whoſe Diameter is 1.

Example V.

123. To find the Length of any Arch of *Archimedes*'s *Spiral**.

Put CA the radius of the generating circle $=b$, and ARA the circumference of it $=a$; alſo, put the ordinate CB$=y$, the length of the required arch CPB$=z$, and a cir- *Fig.* 63.

---

* See how this Curve is generated, *art.* 39. *note.*

circular arch whoſe radius is the ordinate CB$=x$. Then, (as was found in *art.* 39.) $\dot{x}=\frac{ay\dot{y}}{b^2}$, and therefore $\dot{x}^2=\frac{a^2y^2\dot{y}^2}{b^4}$; which ſubſtituted for $\dot{x}^2$, makes the general expreſſion for the Fluxion of the Curve, viz.

$$\dot{z}=\overline{\dot{x}^2+\dot{y}^2}\Big|^{\frac{1}{2}} \text{ (art. 118.) } =\overline{\frac{a^2y^2\dot{y}^2}{b^4}+\dot{y}^2}\Big|^{\frac{1}{2}} =$$

$$\frac{\dot{y}}{b^2}\times\overline{a^2y^4+b^4}\Big|^{\frac{1}{2}} = \frac{\dot{y}\times\overline{a^2y^2+b^4}}{b^4\times\overline{a^2y^2+b^4}\Big|^{\frac{1}{2}}} =$$

$$\frac{a^2y^3\dot{y}+b^4y\dot{y}}{b^2\times\overline{a^2y^4+b^4y^2}\Big|^{\frac{1}{2}}} = \frac{a^2y^3\dot{y}+\frac{1}{2}b^4y\dot{y}}{b^2\times\overline{a^2y^4+b^4y^2}\Big|^{\frac{1}{2}}} +$$

$$\frac{\frac{1}{2}b^4\dot{y}}{b^2\times\overline{a^2y^2+b^4}\Big|^{\frac{1}{2}}} = \frac{1}{b^2}\times\overline{a^2y^4+b^4y^2}\Big|^{-\frac{1}{2}}\times$$

$$\overline{a^2y^3\dot{y}+\tfrac{1}{2}b^4y\dot{y}} + \frac{b^2}{2a}\times\frac{a\dot{y}}{\overline{a^2y^2+b^4}\Big|^{\frac{1}{2}}}.$$ Now, the Fluent of the firſt of theſe two terms is found by *art.* 114. $=\frac{1}{2b^4}\times\overline{a^2y^4+b^4y^2}\Big|^{\frac{1}{2}}$; and the Fluent of the laſt is found by *art.* 115. $=\frac{b^2}{2a}\times$ Hyp. Log. of $ay+\overline{a^2y^2+b^4}\Big|^{\frac{1}{2}}$; therefore $z=\frac{1}{2b^2}\times\overline{a^2y^4+b^4y^2}\Big|^{\frac{1}{2}}+\frac{b^2}{2a}\times$ Hyp. Log. of $ay+\overline{a^2y^2+b^4}\Big|^{\frac{1}{2}}$: but, when z and y are

$=0$,

$=0$, as at the point C, this equation becomes $0 = \frac{b^2}{2a} \times$ Hyp.Log. of b^2; and therefore, the Fluent Corrected, (*art.*117.) makes the *true* value of $z = \frac{1}{2b^2} \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $\overline{ay + \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}} - \frac{b^2}{2a} \times$ Hyp. Log. of $b^2 =$ (by the Nature of Logarithms; the Difference of the Logarithms of any two Numbers being equal to the Logarithm of their Quotient;) $\frac{1}{2b^2} \times \overline{a^2y^4 + b^4y^2}\big|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $\frac{ay + \overline{a^2y^2 + b^4}\big|^{\frac{1}{2}}}{b^2} =$ the Length of the Arch CPB required. And therefore, by ſubſtituting the radius b for the ordinate y, we ſhall have the Length of the whole Spiral CPBA $= \frac{1}{2} \times \overline{a^2 + b^2}\big|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $\frac{a + \overline{a^2 + b^2}\big|^{\frac{1}{2}}}{b}$.

CHAP. IV.

Of finding the Areas of Curvilineal Spaces.

Fig. 64. 65. 124. IN Curves whoſe ordinates are referred to an Axis (*fig.* 64.) let *bc* be conceived indefinitely near and parallel to the perpendicular ordinate BC, and B*n* equal and parallel to C*c* the Increment of the abſciſs AC: then, becauſe *bn* bears no aſſignable ratio to BC, *bc* may be taken as equal to BC or *nc*; and the trapezium BC*cb* as equal to the parallelogram BC*cn*: but, BC*cb* is the Moment or Increment of the curvilineal ſpace ABC, that is, (if we put $AC=x$, $CB=y$, and $Cc=x'$,) the Moment or Increment of the Space ABC is $=yx'$; and therefore, (*art.* 7.) the Fluxion of it is $=y\dot{x}$.——In like manner, in Spirals, or thoſe Curves whoſe ordinates are referred to a fixed or central Point (*fig.* 65.) let *b*C be conceived indefinitely near to the ordinate BC, and the little circular arch B*n* (whoſe radius is CB,) be ſuppoſed a little right line perpendicular to *b*C: then, *bn* having leſs than any aſſignable ratio to *n*C, BC*n* may be conſidered as equal to BC*b* the Moment or Increment of the curvilineal Space BPCB; that is, (if we put $CB=y$, and $Bn=x'$,) the

Moment or Increment of the Space CPBC is $=\frac{1}{2}yx'$, or it's Fluxion $=\frac{1}{2}y\dot{x}$.

Or, Let the curvilineal ſpace AEI and parallelogram AG (*fig.* 64.) be generated by the perpendicular and indefinite right line AF moving with a parallel motion from A along the axis AE; then, it is evident, the curvilineal ſpace will increaſe ſlower, or flow with a leſs degree of velocity, than the parallelogram, before the ſaid generating line arrives at the term CB; and afterwards faſter, or with a greater degree of velocity: therefore, at the ſaid term, they will flow with one and the ſame degree of velocity; that is, at the term CB, the Fluxions of the curvilineal ſpace and parallelogram will be equal: But, it is plain, the Fluxion of the parallelogram, at the term CB, is equal to DA or BC drawn into the Fluxion of AC; therefore, the Fluxion of the curvilineal ſpace ACB is equal to the ordinate CB drawn into the Fluxion of the abſciſs AC; that is, (putting AC$=x$, and CB$=y$,) the Fluxion of the curvilineal Space ACB is $=y\dot{x}$.——In *fig.* 65. let the curvilineal ſpace CPIC be generated by the variable right line CF turning round the center C; and, at the ſame time, let the ſector CDGC be deſcribed

by

by the radius CD; then, it is plain, before the line CF comes to be in the ſituation CB, the ſpiral ſpace will increaſe ſlower, or flow with a leſs degree of velocity, than the circular ſpace or ſector CDG; and afterwards faſter, or with a greater degree of velocity: therefore, at the term CB, they will increaſe or flow with an equal degree of velocity: But, it is evident, the velocity with which the ſector enlarges, is equal to half it's radius drawn into the velocity with which it's arch is deſcribed; therefore, the velocity with which the curvilineal ſpace CPBC is increaſed, at the term CB, is equal to $\frac{1}{2}$ CB drawn into the velocity of the point D or B moving along the arch DG at the point B; that is, the Fluxion of the ſaid curvilineal ſpace is equal to $\frac{1}{2}$ CB drawn into the Fluxion of the circular arch DB; or (putting the ordinate $CB = y$, and arch $DB = x$,) the Fluxion of the curvilineal Space CPBC is $= \frac{1}{2} y\dot{x}$; as before.

125. WHEREFORE, when the Curve is referred to an Axis, (*fig.* 64.) find the value of y in terms of x, by help of the Equation of the given Curve, which multiply by $\dot{x}$; or, find the value of $\dot{x}$ in terms of y, which multiply by y: Then, the Fluent of the re-

ſulting fluxional expreſſion will expreſs the Area (or quadrature) of the curvilineal Space ABC required.——And, when the Curve is referred to a fixed or central Point C, (*fig*.65.) find the value of $\dot{x}$ in terms of $\dot{y}$, from the properties of the given curve; then multiply this value of $\dot{x}$ by $\frac{1}{2}y$, and find the Fluent; and it will give the Area of the Space required.

EXAMPLE I.

126. To find the Area of the Space ABCA; the Curve AB being a *Parabola*. *Fig.* 66.

Put the abſciſs AC$=x$, ordinate CB$=y$, and the parameter $=1$. Then, by the nature of the curve, $x=y^2$, or $x^{\frac{1}{2}}=y$; therefore, $y\dot{x}$ (the Fluxion of the Area, *art.* 124.) $=x^{\frac{1}{2}}\dot{x}$; the Fluent of which is $\frac{2}{3}x^{\frac{3}{2}}$ = (by ſubſtituting y for $x^{\frac{1}{2}}$ it's value,) $\frac{2}{3}xy$ = the Area required.

Or, The Fluxion of the above equation of the curve, viz. of $x=y^2$, is $\dot{x}=2y\dot{y}$; which, multiplied by y, makes the Fluxion of the Area, viz. $y\dot{x}$ (*art.* 124.) $=2y^2\dot{y}$; the Fluent of which is $\frac{2}{3}y^3$ = (by writing x for y^2 it's value,) $\frac{2}{3}xy$; as before.

Corollary.

The Area of any Parabolic Space ABCA is equal to two-third-parts of it's circumſcribing Parallelogram ADBC.

EXAMPLE II.

Fig. 67. 127. To find the Area of the Space ABCG; the property of the Curve AB being ſuch, that it's Subtangent CT is invariable, or always of the ſame value. (See *art.* 80.)

Put the given Subtangent CT$=a$, GA$=b$; GC$=x$, and CB$=y$. Then, (*art.* 25.) $a=\frac{\dot{x}y}{\dot{y}}$; therefore, $\dot{x}=\frac{a\dot{y}}{y}$; which, multiplied by y, makes the Fluxion of the Space viz. $y\dot{x}$ (*art.* 124.) $=a\dot{y}$; and the Fluent of this Fluxion is ay: But, when the area of the ſpace is $=0$, or $y=b$, this expreſſion for the Fluent becomes $=ab$; and therefore, (*art.* 117. *Ex.* 3.) the Fluent corrected is $ay-ab$ $=\overline{y-b}\times a=$ the Area of the Space ABCG required.

EXAMPLE III.

Fig. 63. 128. To find the Area of the Space CPBC; the Curve CPBA being the *Spiral* of *Archimedes* *.

Put the circumference of the generating circle ARA$=a$, and it's radius CA$=b$;

---

* [illegible] generation of this Curve, *art.* 39. *note.*

alſo, put the ordinate CB$=y$. Now, (as was found in *art.* 39.) $\dot{x}=\frac{ay\dot{y}}{b^2}$; which equation multiplied by $\frac{1}{2}y$ makes the Fluxion of the Space viz. $\frac{1}{2}y\dot{x}$ (*art.* 124.) $=\frac{ay\dot{y}}{b^2}\times\frac{1}{2}y$ $=\frac{ay^2\dot{y}}{2b^2}$; the Fluent of which is $\frac{ay^3}{6b^2}=$ the Area of the Space required. And therefore, by ſubſtituting b for y, we have the Area of the whole ſpiral Space CPBAC $=\frac{1}{6}ab$; which, becauſe the area of a circle is equal to it's periphery drawn into half the radius, is $=\frac{1}{3}$ of the Area of the generating Circle.

EXAMPLE IV.

129. To find the Area of the Space CPBC; the Curve CPB being the *Logarithmic Spiral* *. *Fig.* 68.

Put the ordinate CB$=y$; the length of the curve CPB$=z$; a circular arch, whoſe radius is the ordinate CB, $=x$; and let two given quantities, a and b, be to each other in the ratio of y to z. Then, (as was found in *art.* 40.) $\dot{x}=\dot{y}\times\frac{\overline{b^2-a^2}|^{\frac{1}{2}}}{a}$; which, multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (the general expreſſion for the Fluxion of the Area, *art.* 124.) $= \frac{\overline{b^2-a^2}|^{\frac{1}{2}}}{2a} y\dot{y}$; the Fluent of which is $\frac{\overline{b^2-a^2}|^{\frac{1}{2}}}{4a} y^2$ = the Area of the Space CPBC required.

---

* See the generation of this Curve, *art.* 40. *note.*

EXAMPLE V.

Fig. 69. 70. 130. To find the Area of the Space CHBRC; the Curve being a *Spiral* generated by a point moving uniformly along the ſemicircle CDA, from C to A, while the ſaid ſemicircle makes one uniform revolution round the point C as a center *.

Put the radius EC $= a$, arch CRB $= v$, ordinate CB $= y$, and arch DB $= x$. Then (*art.* 41.) $\dot{x} = \frac{4y\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$; which, multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (viz. the Fluxion of the Area, *art.* 124; for any ſpace generated by the arch CRB is equal to that deſcribed by the ordinate CB;) $= \frac{2y^2\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}} = \frac{2y^3\dot{y}}{\overline{4a^2y^2-y^4}|^{\frac{1}{2}}}$ $= \frac{-4a^2y\dot{y}+2y^3\dot{y}}{\overline{4a^2y^2-y^4}|^{\frac{1}{2}}} + \frac{4a^2\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}} =$ (becauſe, *art.*

---

* This Curve was invented *Anno* 1756.

art. 41. $\dot{v} = \frac{2a\dot{y}}{\overline{4a^2-y^2}\big|^{\frac{1}{2}}}$,) $-\overline{4a^2y^2-y^4}\big|^{-\frac{1}{2}} \times \overline{4a^2y\dot{y}-2y^3\dot{y}} + 2a\dot{v}$; the Fluent of which (*art.* 114. 2°.) is $-\overline{4a^2y^2-y^4}\big|^{\frac{1}{2}} + 2av = 2av - y \times \overline{4a^2-y^2}\big|^{\frac{1}{2}} = \mathrm{GC} \times \mathrm{CRB} - \mathrm{CB} \times \mathrm{BG} =$ four times the Area of the ſegment CRBC $=$ the Area required. And therefore, when the point B arrives at G, that is, when $y = 2a$, the Area of the whole ſpiral Space CHBLADC will be equal to four times the Area of the generating Semicircle.

CHAP. V.

Of finding the Convex Superficies of Solids.

LET the ſolid AIV be conceived to be generated by an enlarging or variable circle (whoſe increaſing radius is the variable ordinate of the curve AI,) moving with a parallel motion along the axis from A to E: then will the velocity with which it's convex ſuperficies flows be equal to the periphery of the generating circle drawn into the velocity with which it moves along the curve AI; that is, the Fluxion of the ſaid ſuperficies, *Fig.* 71.

at any term HB, will be equal to the periphery of a circle, whoſe radius is the ordinate CB, drawn into the Fluxion of the curve at the point B.——This follows from *art.* 124. by conſidering the convex ſuperficies as always equal to the area of a curvilineal figure whoſe abſciſs is equal to the curve AH, and the ordinate as equal to the circumference of the generating circle BH whoſe radius is CB.

131. HENCE, if we put the abſciſs AC$=x$, ordinate CB$=y$, curve AB$=z$, and $c=$ 6.28318 &c. $=$ the circumference of a circle whoſe radius is 1 (*art.* 122.); then, becauſe $cy =$ the circumference of a circle whoſe radius is the ordinate CB, and (*art.* 118.) $\dot{z}=\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; the General Expreſſion for the Fluxion of the Convex Superficies of any Solid ABH will be $=cy\dot{z}$ or $cy\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; out of which, by help of the Equation of the given Curve AB, $\dot{x}^2$ or $\dot{y}^2$, *&c.* may be exterminated; and then, the Fluent of the reſulting expreſſion will give the Convex Superficies required: As in the following Examples.

Ex-

EXAMPLE I.

132. To find the Superficies of a *Sphere*, or the Convex Superficies of any Segment of it.

Put the radius EA or EB$=a$, AC$=x$, CB$=y$, and AB$=z$. Let Bn=Cc expreſs the naſcent or very firſt Moment of AC, nb of CB, and Bb of AB; that is, let B$n=x'$, $nb=y'$, and B$b=z'$; then, Bb being conſidered as a little right line coinciding with a tangent to the point B, the triangles ECB and bnB will be alike: (for, $\angle$CB$n=\angle$EBb $=$ a right angle; therefore, $\angle$EBn being common, $\angle$CBE$=\angle n$Bb; and, the angles at C and n being right, the angles CEB and Bbn muſt be likewiſe equal; *ergo*, &c.) Wherefore, by 4 E. 6. EB : BC :: bB : Bn, that is, $a : y :: z' : x'$; $\because z'=\frac{ax'}{y}$, or, (*art.* 7.) $\dot{z}=\frac{a\dot{x}}{y}$; which, ſubſtituted for $\dot{z}$, makes the general expreſſion for the Fluxion of the Convex Superficies viz. $cy\dot{z}$ (*art.* 131.) $=$ $cy\times\frac{a\dot{x}}{y}=ca\dot{x}$; the Fluent of which is $cax=$ the Convex Superficies of the Segment ABH: And therefore, if for x be ſubſtituted $2a$, we

Fig. 72.

ſhall have $2ca^2$ = the Superficies of the whole Sphere.

Corollaries.

1. The Convex Superficies of any Segment of a Sphere is equal to the periphery of a great circle of that Sphere multiplied into the altitude of the Segment.

2. The whole Surface or Superficies of any Sphere is equal to the periphery of it's greateſt circle multiplied into it's diameter, or, equal to the Convex Superficies of it's circumſcribing Cylinder.

EXAMPLE II.

Fig. 133. To find the Convex Superficies of the
73. *Right Cone* AIV, whoſe Altitude AE is given $= a$ and baſe-diameter IV $= b$.

Put AC $= x$, and CB $= y$. Now, the triangles AEI and ACB being alike, by 4 E. 6. AE : EI :: AC : CB; that is, $a : \frac{1}{2}b :: x : y$; $\because x = \frac{2ay}{b}$; the Fluxion of which equation is $\dot{x} = \frac{2a\dot{y}}{b}$; $\because \dot{x}^2 = \frac{4a^2\dot{y}^2}{b^2}$; which, ſubſtituted for $\dot{x}^2$, makes the general expreſſion for the Fluxion of the Convex Superficies viz.

$cy\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ (*art.* 131.) $=cy\times\overline{\frac{4a^2\dot{y}^2}{b^2}+\dot{y}^2}|^{\frac{1}{2}}$ $=\frac{c}{b}\times\overline{4a^2+b^2}|^{\frac{1}{2}}y\dot{y}$; the Fluent of which is $\frac{c}{2b}\times\overline{4a^2+b^2}|^{\frac{1}{2}}y^2=\frac{cy^2}{b}\times\overline{a^2+\frac{1}{4}b^2}|^{\frac{1}{2}}=$ (becauſe by 47 E. 1. $AI=\overline{AE^2+EI^2}|^{\frac{1}{2}}=\overline{a^2+\frac{1}{4}b^2}|^{\frac{1}{2}}$,) $\frac{cy^2}{b}\times AI=$ the Convex Superficies of the Segment ABH: And by writing $\frac{1}{2}b$ for y we have $\frac{1}{4}cb\times IA=$ the Convex Superficies of the Cone AIV.

Corollary.

The Convex Superficies of any Right Cone is equal to half the circumference of it's baſe multiplied into it's ſlant height.

EXAMPLE III.

134. To find the Convex Superficies of the *Parabolic Conoid* ABH. *Fig.* 74.

Put the parameter of the parabola $=a$, AC$=x$, and CB$=y$: then, by the nature of the curve, $ax=y^2$; the Fluxion of which equation is $a\dot{x}=2y\dot{y}$; therefore, $\dot{x}=\frac{2y\dot{y}}{a}$, and

$\dot{x}^2=\frac{4y^2\dot{y}^2}{a^2}$; which, ſubſtituted for $\dot{x}^2$, makes the general expreſſion for the Fluxion of the Convex Superficies viz. $cy\times\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$ (*art.* 131.) $=cy\times\overline{\frac{4y^2\dot{y}^2}{a^2}+\dot{y}^2}|^{\frac{1}{2}}=\frac{c}{a}\times\overline{4y^2+a^2}|^{\frac{1}{2}}y\dot{y}$; the Fluent of which (*art.* 114.) is $\frac{c}{12a}\times\overline{4y^2+a^2}|^{\frac{3}{2}}$: But, at the vertex A, where y vaniſhes or $y=0$, this fluential expreſſion becomes $=\frac{c}{12a}\times\overline{a^2}|^{\frac{3}{2}}=\frac{ca^2}{12}$; therefore, the Fluent corrected (*art.* 117.) is $=\frac{c}{12a}\times\overline{4y^2+a^2}|^{\frac{3}{2}}-\frac{ca^2}{12}=$ the Convex Superficies of the Solid ABH required.

CHAP. VI.

Of finding the Contents of Solids.

Fig. 75. 135. LET *bc* be conceived indefinitely near and parallel to the variable ordinate BC; and B*n* equal and parallel to C*c* the Increment of the variable abſciſs AC. Now, be-

becauſe the little parallelogram BC*cn* is expreſſive of the Increment or Moment of the plane figure ABC, (*art.* 124.) therefore, if the ſolid AIV be conceived to be generated by the revolution of the curvilineal figure AEI round the axis AE, the indefinitely little cylinder generated by the ſaid little parallelogram will expreſs the Moment or Increment of the ſolid at the term BH; and, becauſe this Moment or Increment is equal to the area of the circle deſcribed by the ordinate CB drawn into the Increment of the abſciſs AC, therefore, (*art.* 7.) the Fluxion of the Solid, at the term BH, is equal to the area of a circle, whoſe radius is the ordinate CB, drawn into the Fluxion of the abſciſs AC.

Or, Let the cylinder LG be generated by the parallel motion of the circle LD along the line LN; and, at the ſame time, the ſolid AIV by that of the concentric and variable circle ZF, which, at the vertex A, is ſuppoſed indefinitely ſmall, and continually enlarges as it moves along the curve AI. Then, it is plain, the ſolid AIV will increaſe ſlower, or flow with a leſs degree of velocity, than the cylinder LG, before the generating circles arrive at the term HB; and afterwards faſter, or with a greater degree of velocity: therefore, at the ſaid term, (where the two generating circles become equal, or their peripheries coincide with each other,) they will increaſe, or flow, with the ſame or an equal degree of velocity. But, it is evident, the velocity with which the cylinder flows is equal to the area of it's generating circle drawn into the velocity with which it moves along the line LN; that is, the Fluxion of the cylinder, at the term HB, is equal to the area of a circle, whoſe radius is CB, drawn into the Fluxion of the line LH or AC: Therefore, the Fluxion of the Solid AFV, at the term BH, is equal to the area of a circle, whoſe radius is the ordinate CB, drawn into the Fluxion of the abſciſs AC; as before.

136. HENCE, if we put the abſciſs AC $=x$, ordinate CB $=y$, and $c=3.14159$ &c. $=$ the ſemicircumference of a circle whoſe radius is 1 (*art.* 122.); then, the General Expreſſion for the Fluxion of the Solid Content will be $=cy^2\dot{x}$: out of which, by help of the Equation of the given Curve, $\dot{x}$ or y^2 may be exterminated; and then, by finding the Fluent of the reſulting fluxional expreſſion, we ſhall have the Content of the Solid ABH required.

EXAMPLE I.

137. To find the Content of a *Sphere*, or of any *Segment* of it.

Put AC $=x$, CB $=y$, and the diameter AD $=a$; then, by 35 E. 3. AC×CD = BC× CH, that is, $ax-x^2=y^2$; therefore, by writing $ax-x^2$ for y^2, the general expreſſion for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) becomes $=c\dot{x}\times\overline{ax-x^2}$ $=cax\dot{x}-cx^2\dot{x}$; the Fluent of which is $\frac{cax^2}{2}-\frac{cx^3}{3}=\frac{3cax^2-2cx^3}{6}$ = the Content of the Segment ABH: And therefore, if a be ſubſtituted for x, we ſhall have $\frac{3ca^3-2ca^3}{6}=\frac{1}{6}ca^3$ = the Content of the whole Sphere ABDH. Fig. 76.

Hence, becauſe four times the area of a great circle of the ſphere is $=ca^2$, and the content of a cylinder circumſcribing the ſphere is $=\frac{1}{4}ca^3$, we have the following

Corollary.

The Content of any Sphere is equal to four times the area of it's greateſt circle multiplied into $\frac{1}{6}$th part of it's axis, or, equal to two-third-parts of it's circumſcribing cylinder.

EXAMPLE II.

Fig. 74. 138. To find the Content of the *Parabolic Conoid* ABH, generated by the parabolic ſpace ABC revolving round the axis AC.

Put the parameter $=a$, AC$=x$, and CB $=y$; then, by the nature of the curve, $ax=y^2$. Now, by ſubſtituting ax for y^2, we have the general expreſſion for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $=$ $cax\dot{x}$; the Fluent of which is $\frac{1}{2}cax^2=$ (by writing y^2 for ax,) $\frac{1}{2}cxy^2=$ the Content of the Parabolic Conoid ABH required.

Or, The Fluxion of the equation of the curve, viz. of $ax=y^2$, is $a\dot{x}=2y\dot{y}$; therefore $\dot{x}=\frac{2y\dot{y}}{a}$; which, ſubſtituted for $\dot{x}$, makes the general expreſſion for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $=cy^2\times\frac{2y\dot{y}}{a}=\frac{2cy^3\dot{y}}{a}$; the Fluent of which is $\frac{cy^4}{2a}=$ (by writing ax for y^2,) $\frac{1}{2}cxy^2=$ the Solid Content; as before.

Corol.

Corollary.

The Content of any Parabolic Conoid is equal to half of it's circumſcribing Cylinder.

EXAMPLE III.

139. To find the Content of *any Cone* AIV whoſe baſe is a circle. *Fig.* 77.

Put the given altitude AE$=a$, and diameter VI$=b$. Let HB be parallel to VI; and put AC$=x$, and $c=.78539$ &c. $=$ the area of a circle whoſe diameter is 1. Now, the triangles AVI and AHB being ſimilar, by 4 E. 6. AE : VI :: AC : HB, that is, $a : b :: x : \frac{bx}{a}=$HB; therefore, by 2 E. 12. the area of the circle HB is $=\overline{\frac{bx}{a}}\big|^2 \times c=\frac{cb^2x^2}{a^2}$; which (*art.* 135.) drawn into $\dot{x}$ is $\frac{cb^2x^2\dot{x}}{a^2}=$ the Fluxion of the Content of the Cone at the term CB; the Fluent of which is $\frac{cb^2x^3}{3a^2}$ $=$ the Content of the Cone AHB: And, therefore, by ſubſtituting a for x, we have $\frac{cb^2a^3}{3a^2}=\frac{1}{3}cb^2a=$ the Content of the Cone AVI required.

Or,

Or, Put the altitude AE$=a$, the area of the baſe VI$=b$, and AC$=x$: then, (becauſe ſimilar plane Figures are as the ſquares of their homologous ſides,) we ſhall have $a^2 : b :: x^2 : \frac{bx^2}{a^2}$ = the area of the ſection HB; which (*art.* 135.) drawn into $\dot{x}$ is $\frac{bx^2\dot{x}}{a^2}$ = the Fluxion of the Cone AHB; the Fluent of which expreſſion gives the Content of the ſaid Cone $=\frac{1}{3}\times\frac{bx^3}{a^2}$: And therefore, by writing a for x, we have the Content of the whole Cone AVI$=\frac{1}{3}ab$. Hence, becauſe b may here ſtand for the area of the baſe of any Pyramid whatever, we have the following

Corollary.

The Solid Content of any Cone, or Pyramid, is equal to the area of it's baſe multiplied into one-third-part of it's perpendicular altitude; that is, it is equal to one-third-part of a Cylinder, or Priſm, of the ſame altitude and baſe.

EXAMPLE IV.

140. To find the Content of the Solid ABH, generated by the *Ciſſoidal* Space ABC revolving round the axis AE *. Fig. 78.

Put the axis AE$=a$, abſciſs AC$=x$, and ordinate CB$=y$; then, (as was found in *art.* 34.) $x^3=ay^2-xy^2$; and therefore, $y^2=\frac{x^3}{a-x}$; which, ſubſtituted for y^2, makes the general expreſſion for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) $=\frac{cx^3\dot{x}}{a-x}$ $=\frac{-cx^3\dot{x}}{x-a}=-cx^2\dot{x}-cax\dot{x}-ca^2\dot{x}-\frac{ca^3\dot{x}}{x-a}$ $=-cx^2\dot{x}-cax\dot{x}-ca^2\dot{x}-ca^3\times\frac{-\dot{x}}{a-x}$; the Fluent of which, (becauſe *art.* 21. the Fluxion of the Hyp. Log. of $a-x$ is $\frac{-\dot{x}}{a-x}$,) is $=-\frac{cx^3}{3}-\frac{cax^2}{2}-ca^2x-ca^3\times$ Hyp. Log. of $a-x$. But, when $x=0$, (as at A,) this Fluent becomes $=-ca^3\times$ Hyp. Log. of a: therefore, the Fluent corrected (*art.* 117.) is $=$

---

* See how a *Ciſſoid* is generated, *art.* 34. *note.*

$-\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times$ Hyp. Log. of $\overline{a-x} + ca^3 \times$ Hyp. Log. of $a =$ (becauſe the Difference of the Logarithms of any two numbers is equal to the Logarithm of their Quotient,) $-cx \times \frac{2x^2+3ax+6a^2}{6} + ca^3 \times$ Hyp. Log. of $\frac{a}{a-x} =$ the Content of the Solid ABH required.

Corollary.

When $x = \frac{1}{2}a$, the Content of the Solid will be $= ca^3 \times : -\frac{2}{3} +$ Hyp. Log. of $2 = ca^3 \times 0.02648$ &c. (See *Part* III. *Queſt.* 9.)

PART III.

MISCELLANEOUS QUESTIONS, With their *Incremental* and *Fluxional* SOLUTIONS.

I.

In an *Ellipſis* ABD, whoſe Foci are the points F and K, if a right line B*t* be drawn biſecting the angle FBK; then will the ſaid line be perpendicular to the tangent TBG. *Quære* the Demonſtration? *Fig.* 79.

Let the point *b* be ſuppoſed indefinitely near to B; and with the lines FB and K*b*, as radii, deſcribe the arches B*m* and *bn*: then, if we conſider the ſaid arches as little right lines perpendicular to F*b* and KB reſpectively, (becauſe the Sum of the lines FB and KB is always the ſame invariable quantity, viz. =AD, and therefore the Increment *mb*

=

=the Decrement *n*B,) the right angled triangles B*n*o and *bm*o will be equal and ſimilar, as will therefore the right angled triangles B*mr* and *bnr*; and therefore, if B*b*, the Increment of the curve AB, be ſuppoſed to coincide with the tangent BG, the angles *rb*B and *r*B*b* will be equal, that is; ∠FBT =∠KBG. Therefore, the right line B*t*, which biſects the ∠FBK, makes the ∠*t*BT =∠*t*BG= a Right angle. Q. E. D.

II.

Fig. 80. In an *Hyperbola* AB, whoſe Focus is the point F, and tranſverſe Diameter is DA =KA—AF; the right line *t*B, which biſects the angle KBF, is a tangent to the curve at the point B. *Quære* the Demonſtration?

Suppoſe the point *b* indefinitely near to B, and deſcribe the little arches B*m* and B*n* with the radii KB and FB. Now, becauſe the Difference of the lines KB and FB is always the ſame invariable quantity, viz. = DA, the Increments *mb* and *nb* are equal; and therefore, if the arches B*m* and B*n* be conſidered as little right lines perpendicular to K*b* and F*b* reſpectively, and B*b* (the Increment of the curve AB) as coinciding with the tangent; the little right angled triangles B*mb* and B*nb* will be equal and ſimilar. Therefore, the right line *t*B, biſecting the angle KBF, is a tangent to the curve at the point B. Q. E. D.

III.

In a *Parabola* AB, whoſe Focus is the point F, if a right line KB be drawn parallel to the axis, and the angle KBF be biſected by the right line *t*B; then will this line be a tangent to the curve at the point B. *Quære* the Demonſtration? *Fig.* 81.

Let the indefinite right line LK be perpendicular to LF, and LA=AF; draw *kb* indefinitely near and parallel to KB, and B*m* equal and parallel to K*k*; and with FB, as a radius, deſcribe the little arch B*n*. Now, becauſe the lines KB and FB are always equal to each other, the Increment *mb* is = the Increment *nb*; and therefore, (the indefinitely ſmall arch B*n* being conſidered as a little right line perpendicular to F*b*, and the Increment of the curve, B*b*, as coinciding with the tangent,) the little right angled triangles B*mb* and B*nb* are equal and ſimilar. Conſequently, the right line *t*B, which biſects the angle KBF, is a tangent to the curve at the point B. Q.E.D.

+IV.

Fig. 82. *Quære* the Nature of the Curve APB?—ſuppoſing FP or CE, the diſtance of the parallel and indefinite right lines AC and PE, to be given; the right line AD to paſs through the point P; and $CE \times CD = CB^2$. *

Put $CE = a$, abſciſs $AC = x$, ordinate $CB = y$, and $CD = z$. Let *cd* be ſuppoſed indefinitely near and parallel to CD; and D*m*, B*n*, equal and parallel to C*c*: and put $nb = y'$, $md = z'$; and ſuppoſe TB a tangent to the curve at the point B. Now, by 4 E. 6. DC : CA :: *dm* : *m*D, that is, $z : x :: z' : \frac{xz'}{z} = Dm$ or B*n*; and *bn* : *n*B :: BC : CT, that is, $y' : \frac{xz'}{z} :: y : \frac{xyz'}{zy'} = CT$, or (*art.* 7.) $\frac{xy\dot{z}}{z\dot{y}} = CT$. But, by the queſtion given, $az = y^2$; the Fluxion of which equation is $a\dot{z} = 2y\dot{y}$,

---

* To deſcribe the Curve, or to find the Point B in the line CD through which it muſt paſs.—Produce CD to G, making $DG = FP$; deſcribe the ſemicircle GHC, and draw the perpendicular ordinate DH; laſtly, make $CB = DH$: then will B be the Point required? For, by 35 E. 3. $GD \times DC = DH^2$; that is, $EC \times CD = CB^2$.

$\because \dot{z} = \frac{2y\dot{y}}{a}$; which, ſubſtituted for $\dot{z}$, makes the above value of the Subtangennt CT (viz. $\frac{xy\dot{z}}{z\dot{y}}$) $= \frac{2xy^2}{az} =$ (becauſe $\frac{y^2}{az} = 1$,) $2x$. Therefore, (*art.* 28.) the curve AB is a *Parabola*, whoſe vertex is A.

Corollary.

If F be the Focus; then, by the nature of the Parabola, PF=2FA; and therefore DC=2CA=CT, that is, $z = 2x$.

V.

If TB be a Tangent to the given Curve AB; and another Curve AD be ſo deſcribed as that it's ordinate CD ſhall always be in a given ratio to the correſponding Segment of the former Curve: then, BT will be to TC as the correſponding Segment of the Curve AB is to the Subtangent CV. *Quære* the Demonſtration? *Fig.* 83.

Put CT=s, TB=t, CD=y, AB=z: and let cd be ſuppoſed indefinitely near and parallel to CD; and Dm, Bn, equal and parallel to Cc; that is, let $md = y'$, and B$b = z'$. Now, BT : TC :: bB : Bn, that is, $t : s ::$

$z' : \frac{sz'}{t} = Bn$ or Dm; and $dm : mD :: DC : CV$; that is, $y' : \frac{sz'}{t} :: y : \frac{syz'}{ty'} = CV$, or (*art.* 7.) $\frac{sy\dot{z}}{t\dot{y}} = CV$. Let the given ratio of DC to AB be as a to b, that is, $y : z :: a : b$, $\because z = \frac{by}{a}$; the Fluxion of which equation is $\dot{z} = \frac{b\dot{y}}{a}$; which ſubſtituted for $\dot{z}$ makes the above $\frac{sy\dot{z}}{t\dot{y}} = CV = \frac{bsy}{at}$ (which, by writing z for it's value $\frac{by}{a}$, is) $= \frac{sz}{t}$; therefore, $t : s :: z : CV$; that is, $BT : TC :: AB : CV$. Q. E. D.

VI.

Fig. 84. In the Curve ABD, whoſe Equation (putting the abſciſs $AC = x$, ordinate $CB = y$, and the baſe $AD = a$,) is $ax - x^2 = ay + y^2$: required the Radius of Curvature for any point, and a geometrical Conſtruction to illuſtrate and confirm the Work.

The Fluxion of the given equation of the curve is $a\dot{x} - 2x\dot{x} = a\dot{y} + 2y\dot{y}$; which, making $\dot{x} = 1$, is $a - 2x = a\dot{y} + 2y\dot{y}$;

and the Fluxion of this equation again, the Fluxion of $\dot{y}$ being conſidered as negative, is $-2=-a\ddot{y}+2\dot{y}^2-2y\ddot{y}$. Hence we have $\dot{y}=\frac{a-2x}{a+2y}$, $\dot{y}^2=\frac{a^2-4ax+4x^2}{a^2+4ay+4y^2}$, and $\ddot{y}=\frac{4a^2+8ay+8y^2-8ax+8x^2}{\overline{a+2y}|^3}=$ (becauſe by the equation of the curve, $8ay+8y^2=8ax-8x^2$, or $8ay+8y^2-8ax+8x^2=0$,) $\frac{4a^2}{\overline{a+2y}|^3}$. Now, by ſubſtituting for $\dot{y}^2$ and $\ddot{y}$ theſe their values, in the general expreſſion for the Radius of Curvature, which was found in *art.* 74. to be $=\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when $\dot{x}=1$ and the Fluxion of $\dot{y}$ is negative, we ſhall have

$$\overline{\frac{2a^2+4ay+4y^2-4ax+4x^2}{\overline{a+2y}|^2}}\Bigg|^{\frac{3}{2}}\times\frac{\overline{a+2y}|^3}{4a^2}=$$

(becauſe $4ay+4y^2-4ax+4x^2=0$,) $\overline{\frac{2a^2}{\overline{a+2y}|^2}}\Big|^{\frac{3}{2}}$

$$\times\frac{\overline{a+2y}|^3}{4a^2}=\frac{\overline{2a^2}|^{\frac{3}{2}}}{4a^2}=\frac{\overline{8}|^{\frac{1}{2}}a}{4}=\overline{\tfrac{1}{2}}|^{\frac{1}{2}}a=$$ the Radius of Curvature required; which being a fixed or invariable quantity proves the curve to be an *Arch* of a *Circle*.

Now, if the Radius of a Circle be $\overline{\tfrac{1}{2}}|^{\frac{1}{2}}a$; then, a is the ſide of it's inſcribed ſquare, or the chord of 90°. as is the right line AD: For, if the ſaid right line AD be biſected in F, and the perpendicular FE be drawn $= AF = \tfrac{1}{2}a$; the point E will be the center of the circle; and conſequently, the radius will be AE $(= \overline{EF^2 + FA^2}|^{\frac{1}{2}}) = \overline{\tfrac{1}{2}a^2}|^{\frac{1}{2}} = \overline{\tfrac{1}{2}}|^{\frac{1}{2}}a$. And, that x, in the given equation, muſt flow in the ſaid chord AD, may be thus demonſtrated: Draw the right line BC perpendicular to AD, and let it be produced until it meets the circle's periphery in G; and let HI be drawn equal and parallel to AD: then, it is evident, that, $CK = AH = AD$, by conſtruction; and $KG = CB$, becauſe $AC = HK$, and $AB = HG$; therefore, $CG = AD + CB$: But, by 35 E. 3. $AC \times CD = BC \times CG$, that is, (putting $AD = a$, $AC = x$, and $CB = y$,) $x \times \overline{a - x} = y \times \overline{a + y}$, or $ax - x^2 = ay + y^2$. Therefore, &c.

SCHOLIUM.

From the Conſtruction here given, it appears, that, the Queſtion may be diverſified ſo as to be adapted to any regular Polygon that can be inſcribed in a Circle. We ſee here alſo, a demonſtration of the juſtneſs of the *fluxional* Calculus as made uſe of above.

VII.

Suppoſe the Earth to revolve in a circular orbit round the Sun as it's center; and the Moon to revolve round the Earth in the ſame manner; as alſo, that the planes of their orbits do coincide; and that the diameters of the ſaid orbits are as 340 to 1; and laſtly, that the Moon performs 13.368 revolutions to every ſingle revolution of the Earth. *Quære* the Nature and Deſcription of the Curve generated by the Center of the Moon; or, whether the Curve deſcribed by the Center of the Moon, in one Lunation, be any-where *Convex* towards the Sun?

Let S repreſent the Sun; E, the Earth; *Fig.* 85. Eϵ, an arch of the orbit of the Earth paſſed over by it's center in one lunation of the Moon; the circumference of the circle EAF = the concentric arch Aα: Then, (becauſe 13.368—1=12.368= the number of lunations in a year or one revolution of the Earth, and therefore SA : EA :: 12.368 : 1,) when the Moon is in conjunction with the Sun, the diſtance between the Sun and Moon will be greater than the diſtance or radius SA. Now, the Curve deſcribed by the center of the Moon is the ſame as that deſcribed by a point M (EM being the ſemidiameter of the Moon's orbit,) carried round by the rotation of the circle EAF on the arch Aα: It is therefore of the *Cycloidal* Kind, having a point of Inflection, if *every* Cycloid deſcribed by a point within the generating circle is inflected as well upon a *circular* as upon a *rectilinear* baſe (*art.* 65.). To determine which,

Put SA or $SR=a$, EA or $eR=b$, EM or $em=c$, $Rm=r$, $Rd=s$; and let mC be the Radius of Curvature at any point m, which, it is evident, muſt paſs through the point of contact R. Suppoſe the point n indefinitely near to m: Then, Rr and Rr being the indefinitely ſmall contemporary arches with mn, and conſequently the triangles Rmr and Rnr equal in all reſpects; if we conſider the ſaid little arches Rr and Rr as little right lines perpendicular to the radii er and Sr, we ſhall have the $\angle mRn = \angle rRr =$ (becauſe the angles eRr and SRr added to either ſide of the equation make it two right angles,) $\angle Rer + \angle RSr$. Now, $SR : eR :: \angle Rer : \angle RSr$, and $SR : SR + Re :: \angle Rer : \angle Rer +$
$\angle RSr$,

$\angle RSr$, that is, $a : a+b :: \angle Rer : \angle mRn = \frac{a+b}{a} \angle Rer$. Again, in any triangle, as dmr, if the angles mdr, mrd, and Rmr the complement of the obtuſe angle to two right angles, be indefinitely ſmall, they will be proportional to the oppoſite ſides mr, md, and dr*; that is, $dr : md :: \angle Rmr : \angle mrd$; and $dr - md : dr :: \angle Rmr - \angle mrd : \angle Rmr$, that is, $mR : dR :: \angle Rdr : \angle Rmr$, or, $r : s :: \frac{1}{2} \angle Rer : \angle Rnr = \frac{s}{2r} \angle Rer$. And again, $\angle RCn : \angle RnC :: Rn : RC$, that is, $\angle mRn - \angle Rnr : \angle Rnr :: Rm : RC$, or, $\overline{\frac{a+b}{a} - \frac{s}{2r}} \angle Rer : \frac{s}{2r} \angle Rer :: r : RC = \frac{ars}{2ar+2br-as}$. Conſequently, $mR + RC = mC = \frac{2ar^2+2br^2}{2ar+2br-as} =$

r^2

---

* For, let the triangle be circumſcribed (as in *fig.* 86.) by the circle rmd: then will the arches dm and mr differ infinitely little from their chords dm and mr, which therefore may be taken as equal to them. And, ſince by 20 E. 3. the angle at the center of a circle is double of the angle at the periphery, the arches or chords rm and md are the meaſures of double the angles mdr and mrd reſpectively; or, the angles mdr and mrd are to each other as their oppoſite ſides rm and md: and becauſe by 32 E. 1. the angle Rmr is equal to the ſum of the angles mdr and mrd; therefore the meaſure of it is the ſum of the arches or chords rm and md; which differing infinitely little from the ſide or chord rd, may be conſidered as equal to it. *Ergo*, &c.

Fig. 86.

$\dfrac{r^2}{r-\dfrac{as}{2a+2b}}$ = the Radius of Curvature at any point *m*.

Now, it is evident, that, at the point of Inflection, the Radius of Curvature muſt be Infinite; or, that, on one ſide of the ſaid point, the expreſſion for the radius of curvature muſt be affirmative, and on the other negative; therefore, r muſt be more than $\dfrac{as}{2a+2b}$ on one ſide of the ſaid point, and on the other leſs; and conſequently, *at* the point of Inflection, $r=\dfrac{as}{2a+2b}$; which, ſubſtituted for r, makes ($dm\times mR=$) $rs-r^2=\dfrac{2abs^2+a^2s^2}{\overline{2a+2b}|^2}$ = (becauſe $dm\times mR=fm\times ma=$) b^2-c^2; from which equation we have $s=\dfrac{\overline{2a+2b}\sqrt{b^2-c^2}}{\sqrt{2ab+a^2}}$.——Or, to find r, ſay $2ar+2br=as$, or $s=\dfrac{2ar+2br}{a}$; then, ($dm\times mR=$) $rs-r^2=\dfrac{ar^2+2br^2}{a}$ = ($fm\times ma=$) b^2-c^2; which equation gives r =

$= \sqrt{\frac{ab^2 - ac^2}{a+2b}}$, when the point *m* becomes a point of Inflection.

Now, as *m*R (*r*) muſt, by the nature of the circle, always be greater than *ma*; that is, as $\sqrt{\frac{ab^2 - ac^2}{a+2b}}$ muſt always be more than $b-c$; and conſequently, $\frac{ab^2 - ac^2}{a+2b}$ be more than $\overline{b-c}|^2$, that is, $\frac{ab+ac}{a+2b} \times \overline{b-c}$ be more than $\overline{b-c} \times \overline{b-c}$; therefore, *c* muſt always be more than $\frac{b^2}{a+b}$; that is, EM muſt be *more* than a third proportional to ES and EA in order to have a point of Inflection take place in the Curve: But, in the preſent caſe, ES, EA, and EM, being as 13.368, 1, and $\frac{13.368}{340}$ or .039; therefore, EM is *leſs* than the ſaid third proportional; and conſequently, the Curve M$m\mu$, generated by the Center of the Moon, has *not* a point of Inflection, or, is *no-where* Convex towards the Sun. Q. E. I.

Corollaries.

1. When $r=s$, that is, when the point m coincides with a, or a is the generating point; then, the Radius of Curvature will be $= \frac{2ar^2+2br^2}{ar+2br} = \frac{a+b}{a+2b}2r$; and RC $= \frac{a}{a+2b}r$; by analogy, $a+2b : a :: r :$ RC, that is, SF : SA :: mR : RC.

2. When a is Infinite, and $r=s$; that is, when the baſe becomes a right line, and the point m coincides with a, or the curve is the common Cycloid; the Radius of Curvature will be $=2r$. For then, $2br^2$ and $2br$ will be infinitely little in compariſon of $2ar^2$ and ar, and therefore may be rejected.

Fig. 87. 3. When a is Infinite, that is, when the baſe degenerates into a right line, or the curve is the protracted or interior Cycloid, as in *fig.* 87. the Radius of Curvature will become $=\frac{2ar^2}{2ar-as}=\frac{2r^2}{2r-s}$; and therefore, at the point of Inflection, where the radius of curvature is infinite, $2r=s$, that is, Rm = md; and conſequently the right line Rd is perpendicular to the radius ea, and the point a is in the baſe NM. Whence, to find the point of Inflection, we have the following

Conſtruction, viz. Make PR$=ag$, or AR$=$Pg; draw Re equal and parallel to the radius PO; make $ra=$Ng; draw the right line ea; and, laſtly, make $em=$ON: Then will m be the point of Inflection in the interior Semicycloid $_{\mu}m$M.

VIII.

The *Fluxion* of the Hyperbolic Logarithm of any quantity, is equal to the Fluxion of that quantity, divided by the quantity itſelf. *Quære* the Demonſtration? (See *art.* 21.)

Let YAI be an Hyperbola, whoſe aſymptotes are the perpendicular right lines EZ and ET, and whoſe parameter is AP$=$EP$=1$; draw any ordinate CB parallel to PA: then, (as Writers on *Conics* demonſtrate,) the Space PABC will be the Hyperbolic Logarithm of the line EC; and therefore, the Fluxion of the ſpace PABC will be equal to the Fluxion of the Hyperbolic Logarithm of the line EC. Now, the Fluxion of this ſpace, putting EC$=x$ and CB$=y$, is (by *art.* 124.) $=y\dot{x}$; and, by the known property of the curve, EC : EP :: PA : CB, that is, $x : 1 :: 1 : y=\frac{1}{x}$: therefore $y\dot{x}=\frac{\dot{x}}{x}$; that is, *Fig.* 88.

is, the Fluxion of the ſpace PABC, or, of the Hyperbolic Logarithm of the line EC, is equal to the Fluxion of the ſaid line, divided by the line itſelf. Q. E. D.

N. B. By a Space or Line, is meant it's Numerical Meaſure.

IX.

The Hyperbolic Logarithm of 1 being 0, what is the Hyperbolic Logarithm of 10? (See *art.* 21.)

Fig. 89. Let EZ and ET be the aſymptotes of the rectangular Hyperbola YAI, whoſe parameter is AP=PE=1. Then, the area of the ſpace PABC will be the Hyperbolic Logarithm of $\frac{EC}{EP}$, that is, of EC; the area of the ſpace PA*bc* will be the Hyperbolic Logarithm of $\frac{EP}{Ec}$, that is, of $\frac{1}{Ec}$; and, the area of the ſpace *cb*BC will be the Hyperbolic Logarithm of $\frac{EC}{Ec}$; the right lines or ordinates CB and *cb* being ſuppoſed parallel to the aſymptote EZ. Now, to find theſe areas,

1.°. Put $Pc=x$, and $cb=y$; then, by the known property of the curve, $y=\frac{1}{1-x}$;

which, (*art.* 124.) drawn into $\dot{x}$, is $y\dot{x}=\frac{\dot{x}}{1-x}=$ the Fluxion of P*b*$=$ (by throwing $\frac{\dot{x}}{1-x}$ into a Series, *art.* 102.) $\dot{x}+x\dot{x}+x^2\dot{x}+x^3\dot{x}+x^4\dot{x}+x^5\dot{x}+x^6\dot{x}+x^7\dot{x}+x^8\dot{x}+x^9\dot{x}+x^{10}\dot{x}+x^{11}\dot{x}+$ *&c.* therefore the Fluent of this ſeries, viz. $x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+\frac{1}{6}x^6+\frac{1}{7}x^7+\frac{1}{8}x^8+\frac{1}{9}x^9+\frac{1}{10}x^{10}+\frac{1}{11}x^{11}+\frac{1}{12}x^{12}+$ *&c.* is $=$P*b*.

2°. Put PC$=x$, and CB$=y$; then, $y=\frac{1}{1+x}$, and (*art.* 124.) $y\dot{x}=\frac{\dot{x}}{1+x}=$ the Fluxion of PB$=$ (by throwing $\frac{\dot{x}}{1+x}$ into a Series, *art.* 103.) $\dot{x}-x\dot{x}+x^2\dot{x}-x^3\dot{x}+x^4\dot{x}-x^5\dot{x}+x^6\dot{x}-x^7\dot{x}+x^8\dot{x}-x^9\dot{x}+x^{10}\dot{x}-x^{11}\dot{x}+$ *&c.* therefore the Fluent of this ſeries, viz. $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{1}{6}x^6+\frac{1}{7}x^7-\frac{1}{8}x^8+\frac{1}{9}x^9-\frac{1}{10}x^{10}+\frac{1}{11}x^{11}-\frac{1}{12}x^{12}+$ *&c.* is $=$ PB.

Hence,

If P*c*$=$PC, the area *cb*BC, viz. the ſum of P*b* and PB, will be $=2\times : x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\frac{1}{9}x^9+\frac{1}{11}x^{11}+$ *&c.* And P*b*—PB will be $=x^2+\frac{1}{2}x^4+\frac{1}{3}x^6+\frac{1}{4}x^8+\frac{1}{5}x^{10}+\frac{1}{6}x^{12}+$ *&c.* That is, if E*c*$=.9$, and *c*P$=$PC$=.1$;

then, by ſubſtituting .1 for x, we ſhall have $cB = .2006706954$ &c. and $Pb - PB = .0100503358$ &c. half of which added to half cB is $.1053605156$ &c. $= Pb =$ the Hyp. Log. of $\frac{1}{.9}$. And, if $Ec = .8$, and $cP = PC = x = .2$; then, by writing .2 for x, we have $cB = .4054651081$ &c. = the Hyp. Log. of $\frac{1.2}{.8}$; and $Pb - PB = .0408219945$ &c. half of which ſubtracted from half cB is $.1823215567$ &c. $= PB =$ the Hyp. Log. of $\frac{1.2}{1}$ or 1.2; and this ſubtracted from cB leaves $.2231435513$ &c. $= Pb =$ the Hyp. Log. of $\frac{1}{.8}$. Therefore, by the Nature of Logarithms, the Hyp. Log. of $\frac{1}{.9} \times \frac{1.2}{.8} \times 1.2$, viz. of 2, is $= .1053605156$ &c. $+ .4054651081$ &c. $+ .1823215567$ &c. $= .6931471805$ &c. and, the Hyp. Log. of 2^3, viz. of 8, is $= 3 \times .6931471805$ &c. $= 2.0794415416$ &c. and, the Hyp. Log. of $8 \times \frac{1}{.8}$, viz. of 10, is $= 2.0794415416$ &c. $+ .2231435513$ &c. $= 2.3025850929$ &c. Q. E. I.

X.

+ X.

Let the given right line CA turn uniformly round the point C as a center; and, let a point be ſuppoſed to paſs with an uniform motion, from A, along the right line AF, equal and perpendicular to the ſaid line CA, and ſuch velocity, as to arrive at F at the ſame time that the ſaid lines come to be in their firſt ſituation: then, by this point, will the *Spiral* APBF be deſcribed. *Quære* the Area of any Space AR*a*BPA? *, *Fig.* 90.

Put CA$=$AF$=a$, the circumference of the circle AR*a*A$=b$, *a*B$=v$, arch DB (deſcribed with the ordinate or radius CB,) $=x$, CB$=y$, arch AR*a*$=z$, B*n*$=x'$, and *a*a$=z'$. Now, (ſuppoſing B*n*, the Increment of the arch DB, to be a little right line perpendicular to the radius CB,) if we draw *a*G perpendicular to CB, the Moment or Increment of the ſpace AR*a*BD, viz. *a*B*naa*, will, by 41 E. 1. be $=\frac{1}{2}$GB×B*n*$=$ (becauſe by 8 and 4 E. 6. CB : B*a* :: *a*B : BG, or, $y : v :: v :$ BG$=\frac{v^2}{y}$,) $\frac{v^2x'}{2y}$; therefore, *art.* 7. the

---

* This Curve was invented *Anno* 1756.

Fluxion of the ſaid ſpace, which is $=$ the Fluxion of the ſpace in queſtion, is $=\frac{v^2\dot{x}}{2y}$. But, it is plain from the generation of the Curve, (af) $a : b :: \dot{v} : \dot{z} = \frac{b\dot{v}}{a}$, and (C$a$) $a : y :: \dot{z} : \dot{x} = \frac{y\dot{z}}{a} = \frac{y}{a} \times \frac{b\dot{v}}{a} = \frac{by\dot{v}}{a^2}$; which ſubſtituted for $\dot{x}$ makes the above Fluxion of the ſpace in queſtion $=\frac{bv^2\dot{v}}{2a^2}$; the Fluent of which is $\frac{bv^3}{6a^2}$ $=$ the Area of the Space required: And therefore, by writing a for v, we ſhall have the Area of the whole ſpiral Space ARAFBPA$=\frac{1}{6}ab=\frac{1}{3}$ the Area of the Circle CARaA. Q. E. I.

Or, By 47 E. 1. $v=\overline{y^2-a^2}^{\frac{1}{2}}$; the Fluxion of which equation is $\dot{v}=\frac{y\dot{y}}{\overline{y^2-a^2}^{\frac{1}{2}}}$; therefore $\dot{x}\left(=\frac{by\dot{v}}{a^2}\right)=\frac{b}{a^2}\times\frac{y^2\dot{y}}{\overline{y^2-a^2}^{\frac{1}{2}}}$; which ſubſtituted for $\dot{x}$, makes $\frac{1}{2}y\dot{x}$ viz. the Fluxion of the Area *art.*124.(for CB and C*a*B deſcribe equal Spaces,) $=\frac{b}{2a^2}\times\frac{y^3\dot{y}}{\overline{y^2-a^2}^{\frac{1}{2}}}=\frac{b}{6a^2}\times\frac{3y^5\dot{y}-2a^2y^3\dot{y}}{\overline{y^6-a^2y^4}^{\frac{1}{2}}}+$

$\frac{1}{3}b$

$\frac{4}{3}b \times \frac{y\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$; the Fluent of which expreſſion is $\frac{b}{6a^2} \times \overline{y^6 - a^2y^4}|^{\frac{1}{2}} + \frac{1}{3}b \times \overline{y^2 - a^2}|^{\frac{1}{2}} =$ the Area of the Space ACaBPA. So that, by writing $2a^2$ for y^2, we have the Area of the Space AFBPA $= \frac{2}{3}ab$; from which if we take $\frac{1}{2}ab$ (the Area of the Circle ARA,) there will remain $\frac{1}{6}ab =$ the Area of the whole ſpiral Space; as before.

Corollary.

$\frac{\dot{x}y}{\dot{y}}$ (*art.* 38.) is $= \frac{y}{\dot{y}} \times \frac{by^2\dot{y}}{a^2 \times \overline{y^2 - a^2}|^{\frac{1}{2}}} = \frac{by^3}{a^2 \times \overline{y^2 - a^2}|^{\frac{1}{2}}} = \frac{by^3}{a^2v} =$ the Subtangent CT.

Conſtruction. Through the center C draw the indefinite right line AT perpendicular to the ordinate CB; deſcribe through the points A and B a ſemicircle ABH; make CI = the circumference of the circle ARA, and CK = aB; draw right lines HK and IL, and parallel to IL draw MB; laſtly, produce CB to N, making CN = CM, and draw a right line NT parallel to KH: then will T be the point from which a Tangent to the point B muſt be drawn. For, (AC) $a : y :: y : \frac{y^2}{a} =$ CH, and (LC) a : (CI) b :: (BC) y : CM$=$ $\frac{by}{a} =$CN; and (KC) v : (CH) $\frac{y^2}{a}$ :: (NC) $\frac{by}{a}$: CT$=\frac{by^3}{a^2v}$.

XI.

Fig. 91. *Quære* the Content of the cylindrical Ring ABDR?—the radius EA of the inner circumference being given $=a$; and AD, the diameter of the ring, or of the generating circle, $=b$.

Put any arch LA$=x$, and .78539 &c.$=c$: ſuppoſe Ed indefinitely near to ED; and deſcribe the concentric pricked circle BR, making AB$=$BD$=\frac{1}{2}b$. Then, the Moment of the ring, viz. Ad, will be equal to the area of the generating circle AD drawn into the Increment Bb. Now, EA : Aa :: EB : Bb, that is, $a : x' :: a+\frac{1}{2}b :: x'+\frac{bx'}{2a} =$ Bb; and the area of the circle AD$=b^2c$: therefore, the Moment of the ring is $=b^2c \times \overline{x'+\frac{bx'}{2a}} =$ $b^2cx'+\frac{b^3cx'}{2a}$, or it's Fluxion (*art.* 7.) $=b^2c\dot{x}$

$+$

$+\frac{b^3c\dot{x}}{2a}$; the Fluent of which is $b^2cx+\frac{b^3cx}{2a}$ $=\overline{1+\frac{b}{2a}}\times b^2cx=$ the Content of the Ring from L to A; and therefore, by ſubſtituting $8ac$ for x, we have the Content of the whole Ring $=\overline{1+\frac{b}{2a}}\times 8ab^2c^2=b^2c\times\overline{2a+b}\times 4c=$ the area of the generating circle AD drawn into the circumference of the pricked circle ~~A~~R. Q. E. I.
B

XII.

If a heavy Sphere, whoſe diameter is 4 inches, be let fall into a conical Glaſs $\frac{1}{5}$th full of Water, whoſe diameter is 5 inches and altitude 6; how much of the Sphere will be immerſed in the Water?

Put the altitude VH $=6=a$; radius HF $=2.5=b$; 3.14159 &c. (*art.* 122.) $=c$; AD, the diameter of the ſphere, $=4=d$; and AC, that part of the ſaid diameter under the water, $=x$; then, CD $=d-x$. Now, the capacity of the glaſs is $=\frac{1}{3}ab^2c$; and therefore, by the queſtion, the quantity of water in it is $=\frac{1}{15}ab^2c$. By 35 E. 3. AC $\times$ CD $=$ CB^2, that is, $dx-x^2=$ the ſquare of Fig. 92.

the radius of the *ſection* of the ſphere; therefore the area of the ſaid ſection is $=cdx-cx^2$; which drawn into $\dot{x}$ is $cdx\dot{x}-cx^2\dot{x}=$ the *Fluxion* of the ſegment of the ſphere under the water; the Fluent of which is $\frac{1}{2}cdx^2-\frac{1}{3}cx^3=$ the content of the ſaid ſegment. Hence (ſimilar ſolids being as the cubes of their homologous ſides,) $\frac{1}{3}ab^2c : a^3 :: \frac{1}{15}ab^2c+\frac{1}{2}cdx^2-\frac{1}{3}cx^3 : \frac{1}{5}a^3+\frac{3a^2dx^2}{2b^2}-\frac{a^2x^3}{b^2}=VC^3$. But, $VF=\sqrt{a^2+b^2}$; and $HF : FV :: GE : EV$, that is, $b : \sqrt{a^2+b^2} :: \frac{1}{4}d : \frac{d}{2b}\sqrt{a^2+b^2}=EV$; $\therefore VA\ (=VE-AE)=\frac{d}{2b}\sqrt{a^2+b^2}-\frac{1}{2}d=3.2$, which put $=e$; then $VC=e+x$, and $VC^3=e^3+3e^2x+3ex^2+x^3=$ (becauſe by the above VC^3 is $=$) $\frac{1}{5}a^3+\frac{3a^2dx^2}{2b^2}-\frac{a^2x^3}{b^2}$. Now, this equation produces $\overline{10a^2+10b^2}.x^3+\overline{30b^2e-15a^2d}.x^2+30b^2e^2x=2a^3b^2-10b^2e^3$, that is, $422.5x^3-1560x^2+1920x=652$; which equation divided by 422.5 is $x^3-3.692x^2+4.544x=1.543$; whence x may be found $=.546=$

AC = that Part of the Diameter of the Sphere under the Water. Q. E. I.

SCHOLIUM.

WE might now proceed to the inveſtigation of the Centers of *gravity*, *percuſſion*, and *oſcillation*, and a great variety of other Problems in the various branches of Mathematical and Philoſophical Science: but, this Tract being intended as an *Introduction* only, for theſe Things we muſt refer the Learner to the larger and more extenſive Books on the Subject*; in which, though he may meet with many Difficulties, it is hoped they are not ſuch but he will now be able to ſurmount.

The BOOKS, in *Engliſh*, profeſſedly on the Subject, are,

1. A Treatiſe of Fluxions: or, an Introduction to Mathematical Philoſophy. Containing a full explication of that Method by which the moſt celebrated Geometers of the pre-

---

[*] The Author particularly refers to the Works of his two celebrated Friends, Mr. *Emerſon* and the late Mr. *Simpſon*.

preſent Age have made ſuch vaſt Advances in Mechanical Philoſophy. By CHARLES HAYES, *Gent.*——Folio. 315 Pages. Cuts. 1704.

2. An Inſtitution of Fluxions: Containing the firſt Principles, the Operations, with ſome of the Uſes and Applications of that admirable Method. By HUMPHRY DITTON. ——Octavo. 240 Pages. Cuts. 1706.

N. B. A Second Edition was printed in the Year 1726.

3. The Method of Fluxions, both Direct and Inverſe. The former being a Tranſlation from the celebrated Marquis *De L'Hoſpital*'s Analyſe des Infinements Petits; and the latter ſupply'd by the Tranſlator, E. STONE, F. R. S.——Octavo. 450 Pages. Plates. 1730.

4. The Doctrine of Fluxions, founded on Sir *Iſaac Newton*'s Method, publiſhed by himſelf in his Tracts upon the Quadrature of Curves. By JAMES HODGSON, F. R. S. and Maſter of the Royal Mathematical School in Chriſt's Hoſpital.——Quarto. 452 Pages. Cuts. 1736.

N. B. The Title-Page was reprinted in 1756; and, again, in 1758.

5. The Method of Fluxions and Infinite Series; with its Application to the Geometry of Curve-Lines. By the Inventor Sir *Iſaac Newton*, K^t. late Preſident of the Royal Society. Tranſlated from the Author's Latin Original, not yet made public. To which is ſubjoined, a Perpetual Comment upon the whole Work. By JOHN COLSON, M.A. and F.R.S.——Quarto. 339 Pages. Cuts. 1736.

N. B. The ſame Piece was tranſlated by another Hand; and publiſhed, without a Comment, in the Year 1737. Octavo. 189 Pages. Cuts.

⁂ The Original was written in the Year 1671; but founded on a ſmaller manuſcript Tract compoſed in *November* 1666; in which the great Inventor uſed the ſame Method of noting the Fluxions of variable Quantities as that which he afterwards generally followed, that is, *Pointing*.

6. A Mathematical Treatiſe: Containing a Syſtem of Conic-Sections; with the Doctrine of Fluxions and Fluents, applied to various Subjects; viz. To the finding the Maximums and Minimums of Quantities; Radii of Evolution, Refraction, Reflection; ſuperficial and ſolid Contents of curvilinear Figures; Rectification of Curve-lines; Cen-

ters of Gravity, Oſcillation and Percuſſion: as alſo, to the Reſolution of a ſelect Collection of the moſt uſeful, and many new, Phyſico-Mathematical Problems. By JOHN MULLER.——Quarto. 227 Pages. Plates. 1736.

7. The Doctrine and Application of Fluxions. Containing (beſides what is common on the Subject) a number of *new* Improvements in the Theory; and the Solution of a variety of *new* and very intereſting Problems in different Branches of the Mathematics. By THOMAS SIMPSON, F.R.S.—2 Volumes, Octavo. 576 Pages. Cuts. 1750.

⁂ This Work is, perhaps, not inferior to any on the Subject.

N.B. He publiſhed a *Treatiſe of Fluxions* in the Year 1737. Quarto. 216 Pages. Cuts.

†‡† This great and penetrating Genius was born *Auguſt* the 20th, 1710, and died *May* the 14th, 1761.

8. A Treatiſe of Fluxions. By COLIN MAC LAURIN, A.M. Profeſſor of Mathematics in the Univerſity of Edinburgh, and F.R.S.——2 Volumes, Quarto. 754 Pages. Plates. 1742.

⁂ In this maſterly Work, the Subject is handled agreeable to the Method of reaſoning uſed by the ancient Mathematicians.

††† This celebrated Writer was born in *February* 1698, and died *June* the 14th, 1746.

9. The Doctrine of Fluxions: not only explaining the Elements thereof, but alſo its Application and Uſe in the ſeveral Parts of Mathematics and Natural Philoſophy. By W. EMERSON. The Second Edition, corrected and greatly enlarged.——Octavo. 432 Pages. Plates. 1757.

N. B. The Firſt Edition of this elegant and excellent Work was printed in the Year 1743. Octavo. 300 Pages. Plates.

10. Sir *Iſaac Newton*'s two Treatiſes, of the Quadrature of Curves and Analyſis by Equations of an infinite number of Terms, explained. Containing the Treatiſes themſelves, tranſlated into Engliſh, with a large Commentary. By JOHN STEWART, A. M. Profeſſor of Mathematics in the Mariſhal College and Univerſity of Aberdeen.——Quarto. 479 Pages. Cuts. 1745.

††† The Analyſis by Equations was firſt written before the Year 1669; and the Quadrature

of Curves before the *Year 1676*: But, the laſt Scholium in the Quadratures was added but juſt before the *Tract* was publiſhed, which was by the great *Author* himſelf in the Year 1704. The *Analyſis* was firſt printed in 1711.

11. The Method of Fluxions applied to a ſelect number of uſeful Problems. By NICHOLAS SAUNDERSON, L.L.D. Late Profeſſor of Mathematics in the Univerſity of Cambridge.——Octavo. 309 Pages. Plates. 1756.

12. A Treatiſe of Fluxions. By ISRAEL LYONS, Junior.——Octavo. 269 Pages. Plates. 1758.

FINIS.

ERRATA.

Page 41 line 5 for $ay^2\times$ read ay^2+. Page 77 line 16 for CBE read ABE. Page 93 line ult. for Cyclod read Cycloid. Page 131 line 14 for v read u. Page 139 line 7 for $\frac{4y^2}{a^2}$ read $\frac{4y^4}{a^2}$. Page 185 line 20 for $\frac{2}{3}$ read $\frac{1}{3}$. Page 208 line 17 for C*b*B read C*a*B. Page 211 line 8 for AR read BR.

In ſome Copies are likewiſe the following *Errata*, viz. Page 79 line 5 for B*n* AI read B*n* : AI. Page 91 line 10 for $+y$ read $+y^2$. Page 141 line 12 for $5x^6$ read $25x^6$. Page 151 line 9 for $\frac{m}{n}xn^{-1}$ read $\frac{m}{n}x^{\frac{m}{n}-1}$. Page 188 line 1 for cax. read ca^2x. Page 192 line ult. for CB read CB^2.

⁂ The Author believes, there is not ONE *Preſs Error*, in any Copy, but what he has either corrected with a Pen, or pointed out in the above *Errata*.

JOHN ROWE.

May 1. 1767.

Juſt publiſhed,

1. Mathematical Essays: Being Eſſays on Vulgar and Decimal Arithmetic; in which all the Rules are demonſtrated. The *Second Edition.* By Benjamin Donn.——Octavo. 6s.

Sold by W. Johnſton in Ludgate-ſtreet, and B. Law in Ave-mary-Lane.

2. The Accountant and Geometrician: Containing the Doctrine of Circulating Decimals, Logarithms, Book-keeping, and Plane Geometry. By Benjamin Donn. Octavo. 6s.

Sold by J. Johnſon and B. Davenport, in Pater-noſter-row.

⁂ Mr. Donn's Intention is to preſent the World with a New Courſe of Mathematical Learning.

3. Mathematics. With Eleven Copper-Plates. By the late *Reverend* Mr. William West of *Exeter.* The *Second Edition,* with *Additions.* Reviſed by John Rowe.— Octavo. 2s. 6d.

Sold by J. Johnſon and B. Davenport, in Pater-noſter-row.

Fig. 2. p. 2.

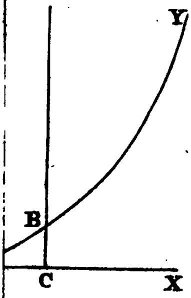


Fig. 3. p. 4.

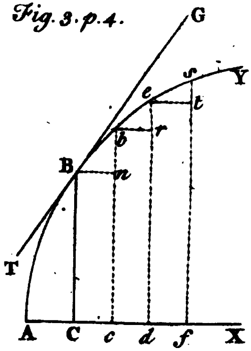


Fig. 5. p. 13.

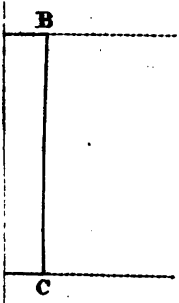
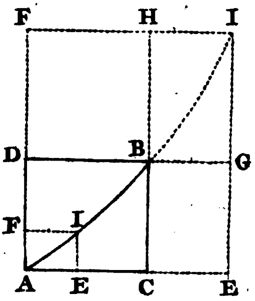


Fig. 6. p. 14.



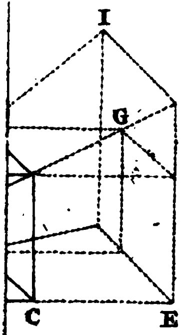
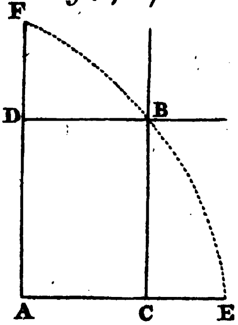


Fig. 8. p. 27.



J. Mynde sc.

o.p. 28.

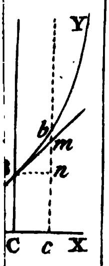


Fig. 11. p. 32.

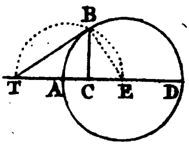
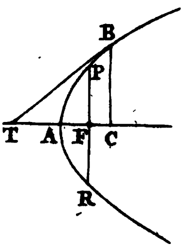


Fig. 12. p. 34.



ig. 14. p. 36.

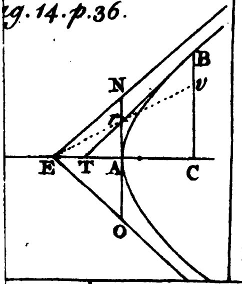


Fig. 15. p. 38.

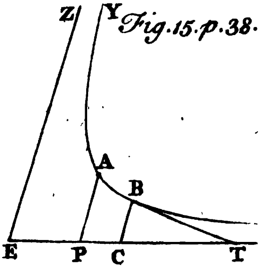
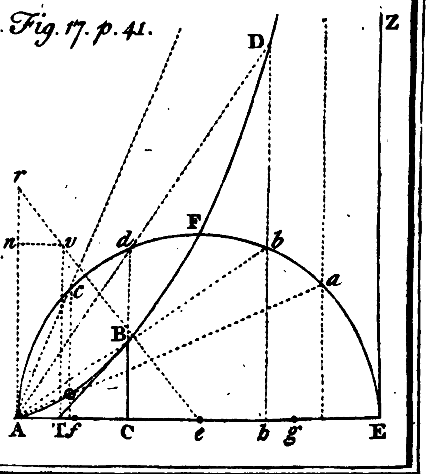


Fig. 17. p. 41.



J. Mynde sc.

Fig: 18 to 24.

Fig. 20. p. 47.

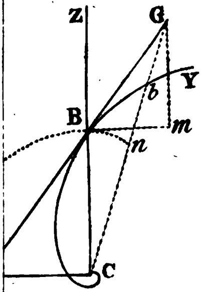


Fig. 22. p. 51.

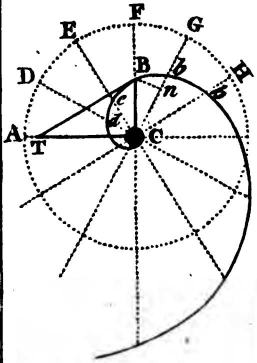


Fig. 21. p. 50.

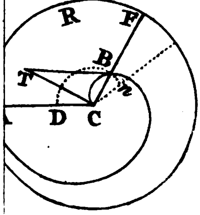
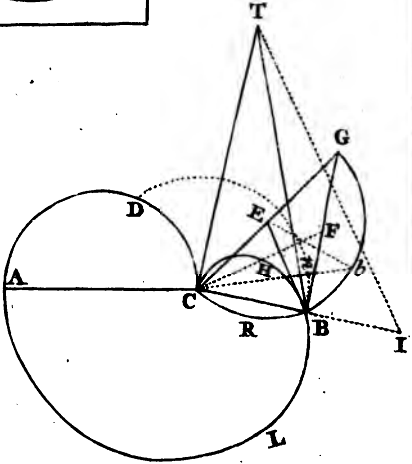


Fig. 24. p. 54.



J. Mynde sc.

Fig: 25 to 34.

Fig. 26. p. 61.

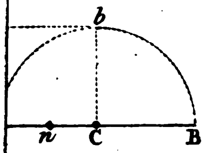
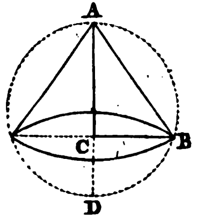


Fig. 27. p. 64.



g. 30. p. 73.

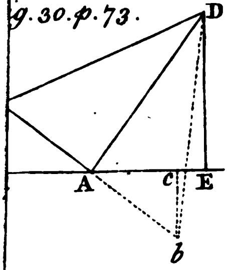
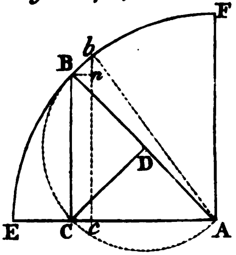


Fig. 31. p. 75.



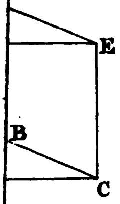
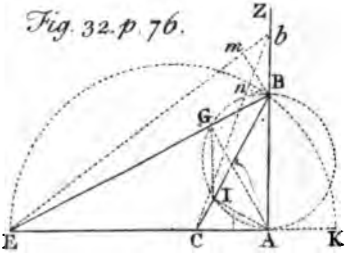


Fig. 32. p. 76.



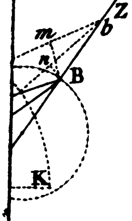
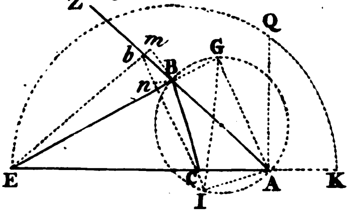


Fig. 34. p. 77.



J. Mynde sc

Fig. 35 to 38.

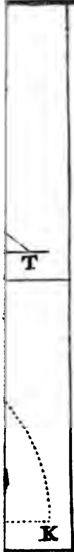
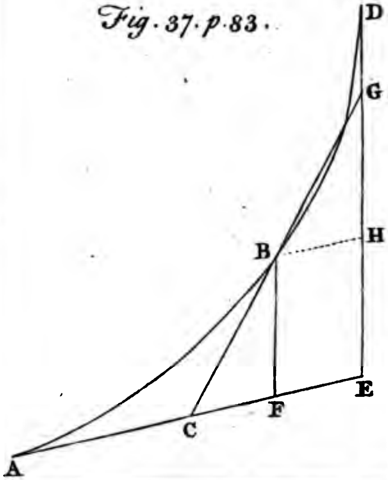
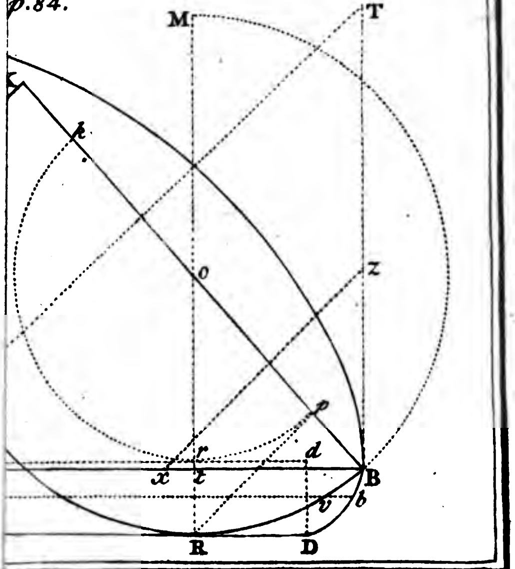


Fig. 37. p. 83.



p. 84.



J. Mynde sc.

Fig. 39. to 43.

Fig. 40. p. 88.

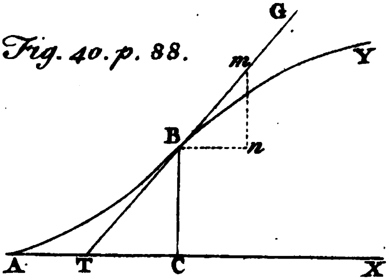


Fig. 41. p. 92.

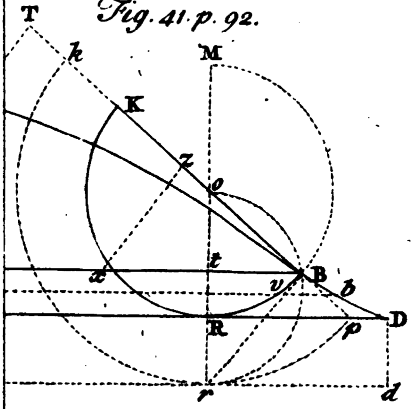
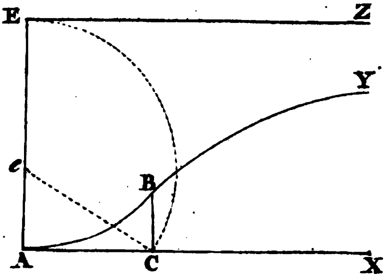


Fig. 43. p. 98.



J. Mynde sc.

Fig 53 to 56.

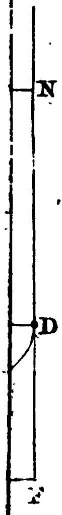


Fig. 54. p. 128.

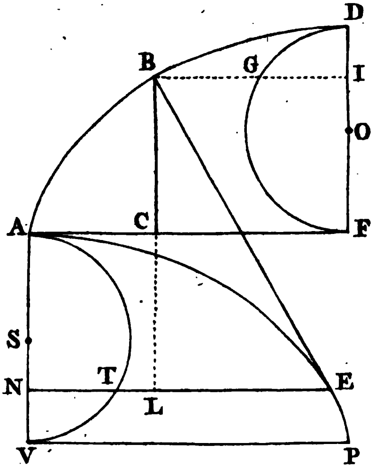
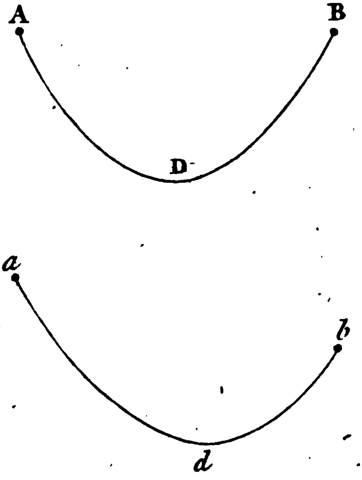


Fig. 56. p. 131.



J. Mynde sc.

Fig. 57 to 70.

Fig. 59. p. 159.

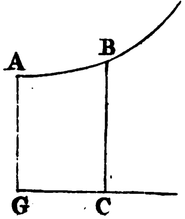


Fig. 60. p. 160.

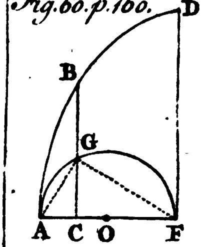


Fig. 63 p. 165. & 172.

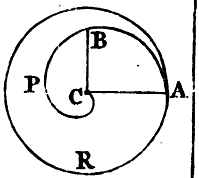


Fig. 64. p. 168.

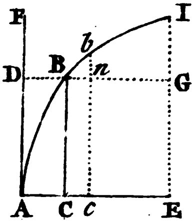


Fig. 67. p. 172.

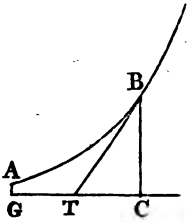


Fig. 68. p. 173.

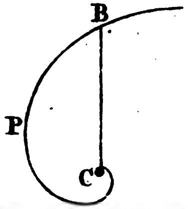
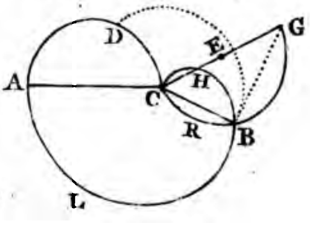


Fig. 70. p. 174.



J. Mynde sc.

72. p. 177.

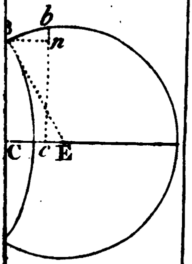
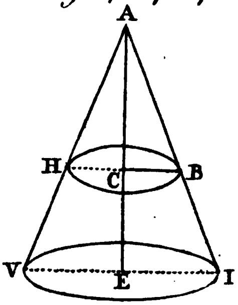


Fig. 73. p. 178.



75. p. 180.

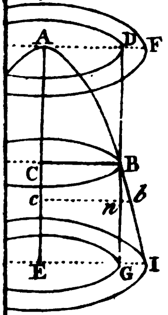


Fig. 77. p. 185.

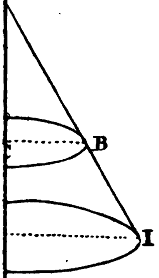
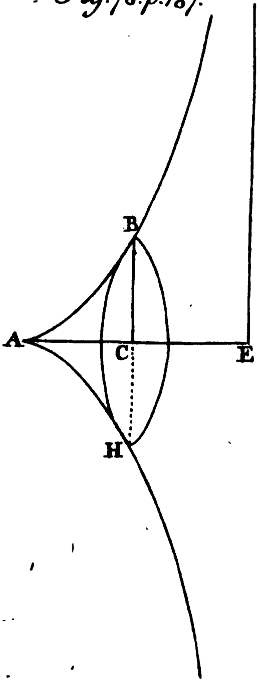


Fig. 78. p. 187.



J. Mynde sc.

Fig: 79. to 84.

Fig. 80. p. 190.

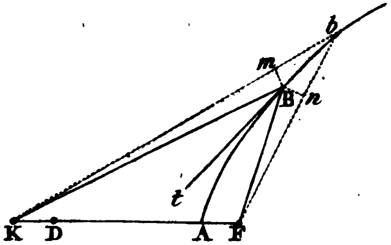


Fig. 82. p. 192.

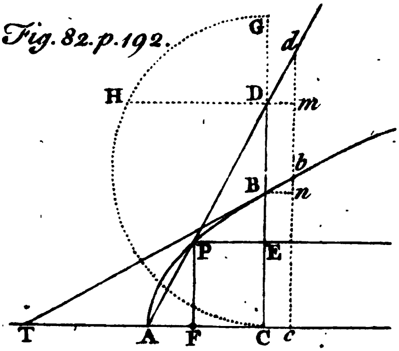
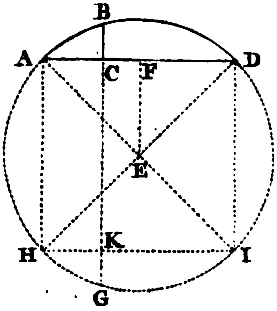


Fig. 84. p. 194.



J. Mynde sc.

Fig. 85. to 87.

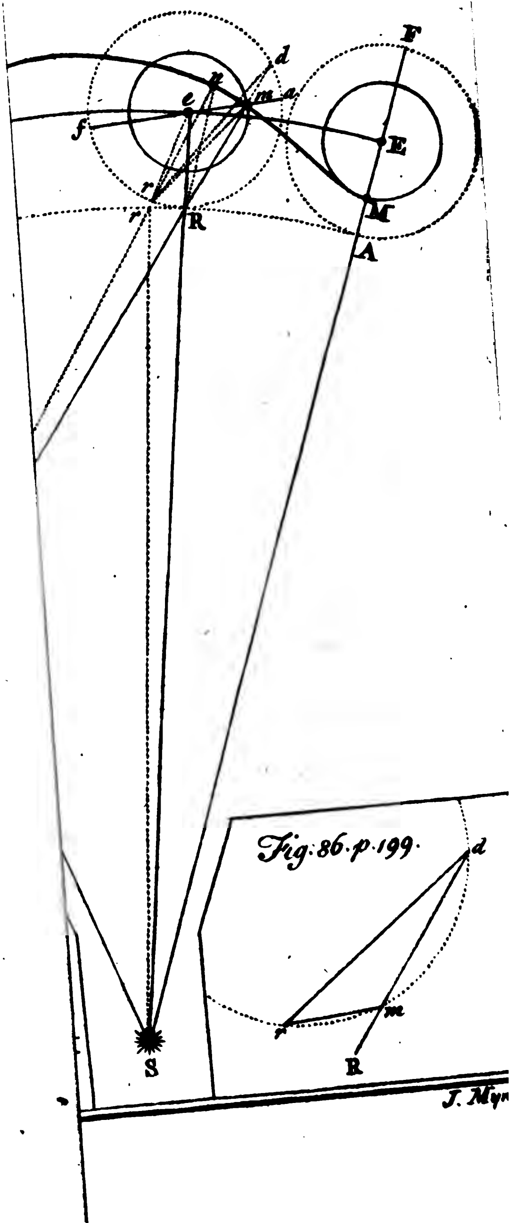


Fig. 86. p. 199.

J. My

Fig. 88. to 92.

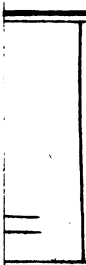
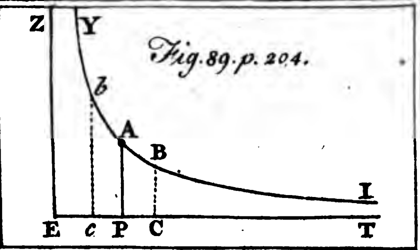


Fig. 89. p. 204.



.207.

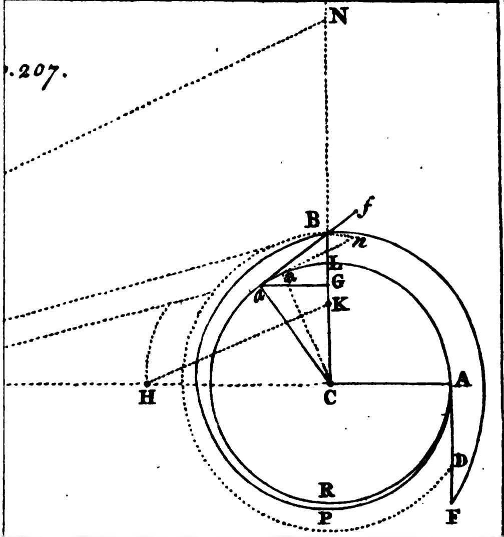
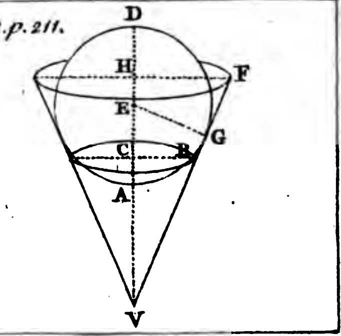


Fig. 92. p. 211.



FINIS.

J. Mynde sc.